Cobases coalgebraically

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Outline

1. Hidden Algebra & Cobases
2. Cobases coalgebraically
3. Proof principle(s) & definition scheme(s) for streams

Motivation

- final coalgebras have several "representations" - how can we understand & use this phenomenon
- definition schemes for stream & stream functions

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Hidden Algebra (simplified)

Hidden specification

A *hidden specification* is a tuple (Σ, E), where

1. Σ is a many-sorted signature containing *hidden* and *visible sorts*,
2. E is a set of equations.

(Σ, E)-algebra

A (Σ, E)-*algebra* is an algebra for the signature Σ that "behaviourally satisfies" the equations in E.

Hidden Stream Algebra (I)

Hidden sort: Stream
Visible sort: $\mathbb{N}$

$$\left.\begin{array}{rcl} \texttt{head} & : & \texttt{Stream} \to \mathbb{N} \\ \texttt{tail} & : & \texttt{Stream} \to \texttt{Stream} \\ \texttt{cons} & : & \mathbb{N} \times \texttt{Stream} \to \texttt{Stream} \\ \texttt{odd} & : & \texttt{Stream} \to \texttt{Stream} \\ \texttt{even} & : & \texttt{Stream} \to \texttt{Stream} \end{array}\right\} \text{operations } (\Sigma_s)$$

Hidden Stream Algebra (I)

$$\left.\begin{array}{rcl}\texttt{head}(\texttt{cons}(N,S)) & = & N \\ \texttt{tail}(\texttt{cons}(N,S)) & = & S \\ \\ \texttt{head}(\texttt{even}(S)) & = & \texttt{head}(S) \\ \texttt{tail}(\texttt{even}(S)) & = & \texttt{odd}(\texttt{tail}(S)) \\ \\ \texttt{head}(\texttt{odd}(S)) & = & \texttt{head}(\texttt{tail}(S)) \\ \texttt{tail}(\texttt{odd}(S)) & = & \texttt{even}(\texttt{tail}(\texttt{tail}(S)))\end{array}\right\} \text{equations } (E_s)$$

Models: (Σ_s, E_s)-algebras

Hidden Congruences for Streams

Definition:
A relation $R \subseteq A_{\texttt{Stream}} \times A_{\texttt{Stream}}$ is a *hidden congruence* if $(\sigma, \tau), (\sigma', \tau') \in R$ implies

(i) $\texttt{head}(\sigma) = \texttt{head}(\tau)$,

(ii) $(\texttt{tail}(\sigma), \texttt{tail}(\tau)) \in R$, and

(iii) for all $n \in \mathbb{N}$, $(\texttt{cons}(n, \sigma) = \texttt{cons}(n, \tau)) \in R$.

(iv) $(\texttt{even}(\sigma), \texttt{even}(\tau)) \in R$,

(v) $(\texttt{odd}(\sigma), \texttt{odd}(\tau)) \in R$,

In other words
If R is behaves as a congruence w.r.t. the operations in Σ_s.

Experiments

Definition

An *experiment* is a term

$$t[\bullet : \texttt{Stream}]$$

of ("visible") sort $\mathbb{N}$ containing one occurrence of a "place-holder" of type $\texttt{Stream}$.

Examples

$\texttt{head}(\bullet), \texttt{head}(\texttt{tail}^n(\bullet)), \texttt{head}(\texttt{cons}(n, \bullet)) \ldots$

Non-example

$\texttt{tail}(\bullet)$ ("outcome not observable")

Behavioural equivalence

Given a (Σ_s, E_s)-algebra A.

Definition

We define a relation $\equiv$ on A by putting

$$\sigma \equiv \tau \quad :\Leftrightarrow \quad t[\sigma] = t[\tau] \quad \text{for all experiments } t[\bullet].$$

If $\sigma \equiv \tau$ we say σ and τ are *behaviourally equivalent*.

Theorem (Roşu)

Behavioural equivalence is the largest hidden congruence.

Coinduction Method

Suppose we want to show $\sigma_1 \equiv \sigma_2$. Then

Step 1. Pick an “appropriate” binary relation R on terms of type `Stream`.

Step 2. Prove that R is a hidden congruence.

Step 3. Show that $(\sigma_1, \sigma_2) \in R$.

Issues

Redundancy

In order to prove that R is a hidden congruence for the stream specification, one has to show that it is a congruence w.r.t. *all* operations.

Alternative representations

We can describe a stream not only using $\{\texttt{head}, \texttt{tail}\}$ but, e.g. , also using $\{\texttt{head}, \texttt{even}, \texttt{odd}\}$.

Some Notation

Notation

For terms $t, t' \in T_{\texttt{Stream}}$ we write

$$(\Sigma_s, E_s) \models t = t'$$

if for all (Σ_s, E_s)-algebras and all valuations $\theta : \mathrm{Var} \to A$:

$$\theta(t) \equiv \theta(t').$$

Cobases (simplified)

Cobasis

A set of operations $\Delta \subseteq \Sigma_s$ is called *cobasis* if for all $t, t' \in T_{\texttt{Stream}}$

$$\frac{(\Sigma_s, E_s) \models \delta(t) = \delta(t') \text{ for all “suitable” } \delta \in \Delta}{(\Sigma_s, E_s) \models t = t'.}$$

Intuition

The operations in Δ are sufficient to *observe* all the relevant information about a given stream.

Examples

Cobases

- Σ_s
- $\{\texttt{head}, \texttt{tail}\}$
- $\{\texttt{head}, \texttt{even}, \texttt{odd}\}$

Non-examples

- $\{\texttt{head}, \texttt{even}\}$
- $\{\texttt{tail}, \texttt{odd}\}$

Hidden algebras as coalgebras

(**simplified:** for a moment we forget about `even` and `odd`)

A given stream algebra A can be written as follows:

$$A \xrightarrow{\langle h,t,\lambda x\lambda n.c(n,x)\rangle} \mathbb{N} \times A \times A^{\mathbb{N}}$$

The final stream algebra:

$$\mathbb{N}^{\omega} \xrightarrow{\langle \mathtt{head},\mathtt{tail},\lambda x\lambda n.\mathtt{cons}(n,x)\rangle} \mathbb{N} \times \mathbb{N}^{\omega} \times (\mathbb{N}^{\omega})^{\mathbb{N}}$$

In which sense final?

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In which sense final?

The final stream (co-)algebra

Let $(\Omega, \langle H, T, C\rangle)$ be the final $\mathbb{N} \times \_ \times (\_)^{\mathbb{N}}$-coalgebra. Furthermore put

$$P := \{t \in \Omega \mid \text{for all } n \in \mathbb{N}.\ H(C(n,t)) = n \ \ \&\ \ T(C(n,t)) = t\}.$$

Let $\Box P$ denote the largest subcoalgebra of $(\Omega, \langle H, T, C\rangle)$ that is contained in P.

Proposition

$(\mathbb{N}^{\omega}, \langle \texttt{head}, \texttt{tail}, \texttt{cons}\rangle) \cong \Box P$.

Corollary

The coalgebra $(\mathbb{N}^{\omega}, \langle \texttt{head}, \texttt{tail}, \texttt{cons}\rangle)$ is final among all $\mathbb{N} \times \_ \times (\_)^{\mathbb{N}}$-coalgebras that satisfy the equations E_s.

From signature to functors

We do only an example:

Consider $\Delta = \{\mathtt{head}, \mathtt{tail}, \mathtt{cons}\}$

$$\begin{aligned} \mathtt{head} &: \mathtt{Stream} \to \mathbb{N} \\ \mathtt{tail} &: \mathtt{Stream} \to \mathtt{Stream} \\ \mathtt{cons} &: \mathtt{Stream} \to (\mathtt{Stream})^{\mathbb{N}} \end{aligned}$$

Functor G_Δ

$$G_\Delta X := \mathbb{N} \times X \times (X)^{\mathbb{N}}$$

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A categorical property of cobases

Observation

$\Delta \subseteq \Sigma_s$ is a cobasis for (Σ_s, E_s) iff $\mathbb{N}^\omega$ together with the operations in Δ is (isomorphic to) a subcoalgebra of the final G_Δ-coalgebra.

Examples

- The set $\{\mathtt{head}, \mathtt{tail}\}$ is a cobasis: $(\mathbb{N}^\omega, \langle \mathtt{head}, \mathtt{tail} \rangle)$ is a final $\mathbb{N} \times \_$-coalgebra.
- The set $\{\mathtt{head}, \mathtt{even}, \mathtt{odd}\}$ is a cobasis: $(\mathbb{N}^\omega, \langle \mathtt{head}, \mathtt{even}, \mathtt{odd} \rangle)$ is isomorphic to a subcoalgebra of the final $\mathbb{N} \times \_ \times \_$-coalgebra.

Complete set of operations

Definition

Let X be a set. A collection of operations $\{f_\sigma\}_{\sigma \in \Sigma}$ for some $\mathcal{S}$-sorted signature Σ is called *complete for X* if the final map $\varphi : X \to \Omega_{G_\Sigma}$ is injective:

$$\begin{array}{ccc} X & \overset{\exists! \varphi}{\hookrightarrow} & \Omega_{G_\Sigma} \\ \downarrow{\scriptstyle \langle f_\sigma : \sigma \in \Sigma \rangle} & & \downarrow{\scriptstyle \omega_\Sigma} \\ G_\Sigma X & \underset{G_\Sigma \varphi}{\hookrightarrow} & G_\Sigma \Omega_{G_\Sigma} \end{array}$$

Equivalently

... if $(X, \langle f_\sigma : \sigma \in \Sigma \rangle)$ is isomorphic to a subcoalgbra of the final coalgebra.

More abstractly: Complete coalgebra

Definition

Let X be a set and let $G : \mathsf{Set} \to \mathsf{Set}$ be a functor. Then $\alpha : X \to GX$ is called *complete* for X if the final G-coalgebra map $\varphi : X \to \Omega_G$ satisfies $\ker(\varphi) \subseteq \Delta_X$.

$$
\begin{array}{ccc}
X & \overset{\exists! \varphi}{\hookrightarrow} & \Omega_G \\
\alpha \downarrow & & \downarrow \omega_\Sigma \\
GX & \underset{G\varphi}{\hookrightarrow} & G\Omega_G
\end{array}
$$

Equivalently

... if (X, α) is isomorphic to a subcoalgbra of the final coalgebra.

Complete sets of operations

For streams

- the examples from the beginning
- the set $\{\texttt{head}, \texttt{even}', \texttt{odd}'\}$ where

$$\begin{aligned} \texttt{even}'(a_0a_1a_2a_3a_4\ldots) &= a_2a_4\ldots \\ \texttt{even}'(a_0a_1a_2a_3a_4\ldots) &= a_1a_3\ldots \end{aligned}$$

- the set $\{\texttt{head}, \texttt{tail}^{\mathbb{N}}\}$ where $\texttt{tail}^{\mathbb{N}}(a_0a_1a_2\ldots) = (1 * a_1, 2 * a_2, 3 * a_3, \ldots)$.

Other examples

- bi-infinite streams: $\{\texttt{head}, \texttt{ltail}, \texttt{rtail}\}$
- binary trees: $\{\texttt{head}, \texttt{left}, \texttt{right}\}$

Proofs & Definitions

Idea

Given a complete set of operations for a set X we obtain a proof principle and a definition principle for X.

For now

We focus on streams & stream functions and the complete set of operations $\Delta = \{\mathsf{head}, \mathsf{even}, \mathsf{odd}\}$.

Proof principle

We can use complete sets Δ of operations, in order to prove things:

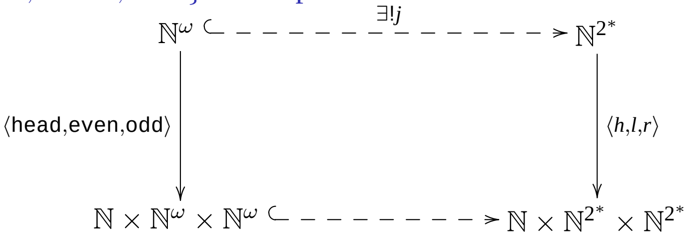
- using Δ-coinduction:
  $\mathtt{head}(\sigma) = \mathtt{head}(\tau), \mathtt{even}(\sigma) = \mathtt{even}(\tau)$ and $\mathtt{odd}(\sigma) = \mathtt{odd}(\tau)$ implies $\sigma = \tau$ (e.g. show $\mathtt{zip}(\mathtt{even}(\sigma), \mathtt{odd}(\sigma)) = \sigma$).
- using Δ-bisimulations:
  R is a Δ-bisimulation if for all $(\sigma, \tau) \in R$ we have $\mathtt{head}(\sigma) = \mathtt{head}(\tau)$, $(\mathtt{even}(\sigma), \mathtt{even}(\tau)) \in R$ and $(\mathtt{odd}(\sigma), \mathtt{odd}(\tau)) \in R$.

Definition priniciple for $\{\mathsf{head}, \mathsf{even}, \mathsf{odd}\}$

Definition

Let $\mathbb{N}^{2^*} := \{t \mid t : 2^* \to \mathbb{N}\}$ be the set of infinite binary $\mathbb{N}$-labelled trees.

$\{\mathsf{head}, \mathsf{even}, \mathsf{odd}\}$ is complete:



A universal property

$$P := \{t \in \mathbb{N}^{2^*} \mid h(t) = h(l(t))\} \subseteq \mathbb{N}^{2^*}$$

Proposition

We have $(\mathbb{N}^\omega, \langle \mathsf{head}, \mathsf{even}, \mathsf{odd} \rangle) \cong \Box P$.

Corollary

For any coalgebra $(X, \langle H, E, O \rangle)$ with

$$H(x) = H(E(x)) \quad \text{for all } x \in X$$

there is a unique coalgebra morphism $f : X \to \mathbb{N}^\omega$ from $(X, \langle H, E, O \rangle)$ to $(\mathbb{N}^\omega, \langle \mathsf{head}, \mathsf{even}, \mathsf{odd} \rangle)$.

Definition scheme (preparations)

We inductively define the set $\mathcal{FT}$ of flat equation terms and the set $\mathcal{ET}$ of equation terms:

$$\begin{aligned} \mathcal{FT} \ni s &::= x_i \mid E(x_i) \mid O(x_i) \\ \mathcal{ET} \ni t &::= s \in \mathcal{FT} \mid \underline{\tau}, \tau \in A^{\omega} \mid f(t_1, \ldots, t_{r(f)}). \end{aligned}$$

Definition scheme

A *well-formed* sytem of behavioural differential equations $\mathcal{E}$ for a set of function symbols Γ contains for every $f \in \Gamma$ three equations

$$\begin{aligned} H(f(x_1,\dots,x_{r(f)})) &:= c^f(H(x_1),\dots,H(x_{r(f)})) \\ &\qquad \text{for some function } c : \mathbb{N}^{r(f)} \to \mathbb{N} \\ E(f(x_1,\dots,x_{r(f)})) &:= t_E^f(x_1,\dots,x_{r(f)}) \\ O(f(x_1,\dots,x_{r(f)})) &:= t_O^f(x_1,\dots,x_{r(f)}) \end{aligned}$$

where t_E^f and t_O^f are equation terms with free variables contained in $\{x_1,\dots,x_{r(f)}\}$. Furthermore we require that for all $f \in \Gamma$

$$\mathcal{E} \cup \{H(E(x_i)) = H(x_i) \quad | \; x_i \in X\} \vdash$$

$$H(E(f(x_1,\dots,x_{r(f)}))) = H(f(x_1,\dots,x_{r(f)})).$$

Examples

Thue-Morse sequence (in 2^ω)

$\mathtt{TM}_n$ is the binary digit sum of n modulo 2

$$\begin{aligned} H(\mathtt{inv}(x)) &:= 1 - H(x) & \quad H(\mathtt{TM}) &:= 0 \\ E(\mathtt{inv}(x)) &:= \mathtt{inv}(E(x)) & E(\mathtt{TM}) &:= \mathtt{TM} \\ O(\mathtt{inv}(x)) &:= \mathtt{inv}(O(x)) & O(\mathtt{TM}) &:= \mathtt{inv}(\mathtt{TM}) \end{aligned}$$

An example in $\mathbb{Z}^\omega$

$$\begin{aligned} H(\mathtt{alt}(x)) &:= H(x) & \quad H(\mathtt{m}(x)) &:= -H(x) \\ E(\mathtt{alt}(x)) &:= E(x) & E(\mathtt{m}(x)) &:= \mathtt{m}(E(x)) \\ O(\mathtt{alt}(x)) &:= \mathtt{m}(O(x)) & O(\mathtt{m}(x)) &:= \mathtt{m}(O(x)) \end{aligned}$$

Hence $\mathtt{alt}(a_0, a_1, a_2, a_3, \ldots) := a_0,\ -a_1,\ a_2,\ -a_3, \ldots$.

Conclusions

Positive

- coalgebraic understanding of (simple) cobases
- proof principle & definition scheme that works also for structures that cannot be modelled as (final) coalgebras

Many questions...

- Find better & more examples.
- Formulate the definition scheme as general as possible.
- Exisiting work on Hidden Algebra/CoCasL