Pre-Galois Connection on Coalgebras for Generic Component Refinement

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Outline

Motivation

- Component based software development as a promising paradigm to deal with the increasing complexity in software design.
- Components must be specified and implemented before it can be analyzed and used.
- Coalgebras can be used as a mathematical model for components.
- Galois connection has been widely used to ensure the correctness of refinement relations.
- Do we have a notion like Galois connections between coalgebras to witness refinement of components?

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What will we show?

We will show how to. . .

- . . . unify the behavior model and transition types into one functor over the Kleisli category for the coalgebra model of components
- . . . rebuild refinement relationship between coalgebraic structures
- . . . use pre-Galois connection in reasoning about refinement of components

Generic Components

Components can be specified in a generic way which means that the underlying behavior model is taken as a specification parameter, and abstracted to a monad B.

Some useful possibilities:

- *Identity*, $\mathsf{B} = \mathsf{Id}$, which retrieves the simple total and deterministic behavior.
- *Partiality*, $\mathsf{B} = \mathsf{Id} + \mathbf{1}$, i.e., the maybe monad, capturing the partial behavior which describes the possibility of deadlock or failure.
- *Non-determinism*, $\mathsf{B} = \mathscr{P}$, modeling the non-deterministic branching behavior.

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Generic Components

The type of state transitions of the component is described by a functor T. For example, if we take I and O be sets acting as component input and output interfaces, then T can be defined as the **Set** endofunctor

$$\mathsf{T} = (\mathsf{Id} \times O)^I$$

A state-based component can be modeled as a pointed coalgebra $(u \in U, \alpha : U \to \mathsf{BT}U)$ in **Set** with

- B a monad,
- T a functor,
- a distributive law TB $\Rightarrow$ BT implicit, that describes the way how B's effect is distributed over the transition type represented by T,
- the point u being taken as the "initial" or "seed" state.

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Kleisli Category

- For each monad B on **Set**, the Kleisli category for B, denoted by $\mathscr{K}(\mathrm{B})$, can be constructed as follows:
  - Objects in $\mathscr{K}(\mathrm{B})$ are the same as in **Set**. They are just sets.
  - An arrow $U \to V$ in $\mathscr{K}(\mathrm{B})$ is a function $U \to \mathrm{B}V$ in **Set**.
  - Composition of arrows in $\mathscr{K}(\mathrm{B})$ is defined using multiplication $\mu_U : \mathrm{BB}U \to \mathrm{B}U$.
  - Identity arrow $\mathrm{id} : U \to U$ in $\mathscr{K}(\mathrm{B})$ is the unit $\eta_U : U \to \mathrm{B}U$ in **Set**.
- The functor T can be lifted to a functor $\mathscr{K}(\mathrm{T})$ on the Kleisli category $\mathscr{K}(\mathrm{B})$ via the distributive law.
- Considering the component model, a component is just a pointed coalgebra $(u \in U, \alpha : U \to \mathscr{K}(\mathrm{T})U)$ in the Kleisli category $\mathscr{K}(\mathrm{B})$.

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Order

- For a Kleisli category $\mathscr{K}(\mathsf{B})$ and any functor T, an order $\leq_{\mathscr{K}(\mathsf{T})}$ on $\mathscr{K}(\mathsf{T})$ is defined as a collection of preorders $\leq_{\mathsf{BT}U} \subseteq \mathsf{BT}U \times \mathsf{BT}U$, for each set U, such that the following diagram commutes:

$$\begin{array}{ccc} & \overset{\leq_{\mathscr{K}(\mathsf{T})}}{\nearrow} & \textbf{PreOrd} \\ & & \downarrow \\ \mathscr{K}(\mathsf{B}) & \xrightarrow[\mathscr{K}(\mathsf{T})]{} & \mathscr{K}(\mathsf{B}) \end{array} \qquad \begin{array}{ccc} & \nearrow & (\mathsf{BT}U, \leq_{\mathsf{BT}U}) \\ & & \downarrow \\ U & \longrightarrow & \mathsf{BT}U \end{array}$$

- and for any $f : U \to V$, $\mathscr{K}(\mathsf{T})f$ preserves the order, i.e.,

$$u_1 \leq_U u_2 \;\Rightarrow\; \mathscr{K}(\mathsf{T})f(u_1) \leq_{\mathsf{BT}V} \mathscr{K}(\mathsf{T})f(u_2)$$

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Some Possible Examples

- The first example is:

$$x \subseteq_{\mathrm{Id}} y \ \text{ iff } \ x = y$$
$$x \subseteq_{\mathscr{P}} y \ \text{ iff } \ \forall_{e \in x} \exists_{e' \in y} \ .e \subseteq_{\mathrm{Id}} e'$$

The order $\subseteq_{\mathscr{P}}$ captures the classical notion of nondeterministic reduction and can be turned into more specific cases. For example, the failure forcing variant $\subseteq^{E}_{\mathscr{P}}$, where E stands for emptyset, guarantees that the first component fails no more than the second one. It is defined by replacing the clause for $\subseteq_{\mathscr{P}}$ by

$$x \subseteq^{E}_{\mathscr{P}} y \ \text{ iff } \ (x = \emptyset \Rightarrow y = \emptyset) \ \wedge \ \forall_{e \in x} \exists_{e' \in y} \ .e \subseteq_{\mathrm{Id}} e'$$

- Consider the partiality monad $\mathrm{B} = \mathrm{Id} + \mathbf{1}$. The set $\mathrm{B}U$ carry the familiar “flat” order:

$$x \subseteq_{\mathrm{B}} y \ \text{ iff } \ x \neq * \Rightarrow x = y \ \wedge x = * \Rightarrow y = *$$

Forward and Backward Morphisms

- A possible (and intuitive) way of considering component p as a refinement of another component q is to consider that p-transitions are simply preserved in q. For example, for non-deterministic components this means set inclusion.
- Homomorphism can be used to relate two coalgebras.

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\begin{array}{ccc}
U & \xrightarrow{\alpha} & \mathscr{K}(\mathsf{T})U \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \mathscr{K}(\mathsf{T})h} \\
V & \xrightarrow{\beta} & \mathscr{K}(\mathsf{T})V
\end{array}
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- From homomorphisms we can only derive bisimulations!
- To build a witness for refinement relations, we separate the preservation and reflection aspects in homomorphism.

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Forward and Backward Morphisms

For a Kleisli category $\mathscr{K}(\mathrm{B})$ and two coalgebras $p = (U, \alpha : U \to \mathscr{K}(\mathrm{T})U)$ and $q = (V, \beta : V \to \mathscr{K}(\mathrm{T})V)$. A forward morphism $h : p \to q$ with respect to an order $\leq$ on $\mathscr{K}(\mathrm{T})$ is an arrow $h : U \to V$ such that

$$\mathscr{K}(\mathrm{T})h \cdot \alpha \;\leq\; \beta \cdot h$$

Dually, h is called a backward morphism if the following conditions are satisfied:

$$\beta \cdot h \;\leq\; \mathscr{K}(\mathrm{T})h \cdot \alpha$$

Component Refinement

The existence of a forward (backward) morphism connecting two components p and q witnesses a refinement situation whose symmetric closure coincides, as expected, with bisimulation.

Component p is a behavior refinement of q, written $p \sqsubseteq_B q$, if there exist components r and s such that $p \sim r$, $q \sim s$ and a (seed preserving) forward morphism from r to s.

A forward morphism is a “behavior preserving” mapping, but lying inside it is a more fundamental concept: to relate two coalgebras, one must show that all the transitions in one coalgebra are “mimicked” by the other. Such an intuition is formalized by the notion of *simulation*.

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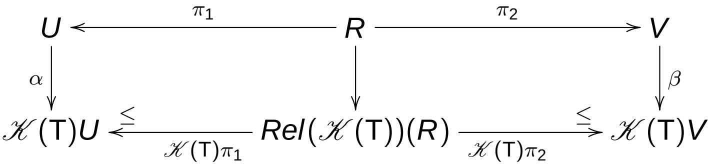
Simulation

- For a given Kleisli category $\mathscr{K}(\mathrm{B})$, a functor T and a refinement preorder $\leq$, a lax relation lifting is an operation $Rel_{\leq}(\mathscr{K}(\mathrm{T}))$ mapping relation R to $\leq \cdot Rel(\mathscr{K}(\mathrm{T}))(R) \cdot \leq$, where $Rel(\mathscr{K}(\mathrm{T}))(R)$ is the lifting of R to $\mathscr{K}(\mathrm{T})$ defined, as usual, as the $\mathscr{K}(\mathrm{T})$-image of inclusion.
- Given coalgebras (U, α) and (V, β), a simulation is a $Rel_{\leq}(\mathscr{K}(\mathrm{T}))$-coalgebra over α and β, i.e., a relation R such that, for all $u \in U, v \in V$,

$$(u, v) \in R \;\Rightarrow\; (\alpha\, u, \beta\, v) \in Rel_{\leq}(\mathscr{K}(\mathrm{T}))(R)$$



Soundness and Completeness Results

For two coalgebras p and q,

Theorem (soundness)

To prove $p \sqsubseteq_B q$ it is sufficient to exhibit a simulation R relating p and q.

Theorem (completeness)

If $p \sqsubseteq_B q$ and h is the witness forward morphism, then $\sim \cdot Graph(h) \cdot \sim$ is a simulation between p and q.

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Pre-Galois Connection

For a Kleisli category $\mathscr{K}(\mathrm{B})$ and functor T, let $\leq$ be an order on $\mathscr{K}(\mathrm{T})$, a pre-Galois connection between two $\mathscr{K}(\mathrm{T})$-coalgebras (U, α) and (V, β) is a pair of arrows $f : U \to V$ and $g : V \to U$, such that for all $u \in U$ and $v \in V$,

$$\alpha(u) \leq_{\mathscr{K}(\mathrm{T})U} \mathscr{K}(\mathrm{T})g \cdot \beta(v) \text{ iff } \mathscr{K}(\mathrm{T})f \cdot \alpha(u) \leq_{\mathscr{K}(\mathrm{T})V} \beta(v)$$

We say that f is the lower adjoint and g is the upper adjoint of the pre-Galois connection.

Composition and Identity for Pre-Galois Connections

- If (f, g) is a pre-Galois connection between two coalgebras $(U, \alpha : U \to \mathscr{K}(\mathrm{T})U)$ and $(V, \beta : V \to \mathscr{K}(\mathrm{T})V)$, and (h, k) is a pre-Galois connection between two coalgebras $(V, \beta : V \to \mathscr{K}(\mathrm{T})V)$ and $(W, \gamma : W \to \mathscr{K}(\mathrm{T})W)$, then $(h \cdot f, g \cdot k)$ is a pre-Galois connection between $(U, \alpha : U \to \mathscr{K}(\mathrm{T})U)$ and $(W, \gamma : W \to \mathscr{K}(\mathrm{T})W)$.
- $(\mathrm{id}, \mathrm{id})$ where id denotes the identity function on U is a pre-Galois connection between a coalgebra $(U, \alpha : U \to \mathscr{K}(\mathrm{T})U)$ and itself.

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Cancellation

If we introduce an order $\lessdot_U$ on U for $(U, \alpha : U \to \mathscr{K}(\mathsf{T})U)$ as $u \lessdot_U u'$ iff $\alpha(u) \leq_{\mathscr{K}(\mathsf{T})U} \alpha(u')$, i.e., we assume that $\lessdot$ reflects the transition structure $\to$. In other words, the functor $\mathscr{K}(\mathsf{T})$ is order-preserving, then

Lemma (Cancellation)

If (f, g) is a pre-Galois connection between two coalgebras $(U, \alpha : U \to \mathscr{K}(\mathsf{T})U)$ and $(V, \beta : V \to \mathscr{K}(\mathsf{T})V)$, then we have

$$f \cdot g \lessdot_V \mathrm{id}_V \text{ and } \mathrm{id}_U \lessdot_U g \cdot f$$

and

Lemma

If (f, g) is a pre-Galois connection between two coalgebras $(U, \alpha : U \to \mathscr{K}(\mathsf{T})U)$ and $(V, \beta : V \to \mathscr{K}(\mathsf{T})V)$, then f and g are both monotonic with respect to $\lessdot_U$ and $\lessdot_V$.

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Relationship with Galois Connection

Theorem

If (f, g) is a pre-Galois connection between two coalgebras $(U, \alpha : U \rightarrow \mathscr{K}(\mathrm{T})U)$ and $(V, \beta : V \rightarrow \mathscr{K}(\mathrm{T})V)$, for the orders $\lessdot_U$ and $\lessdot_V$ on U and V, (f, g) is a Galois connection.

Two-way Similarity

Theorem

If (f, g) is a pre-Galois connection between two coalgebras $(U, \alpha : U \rightarrow \mathscr{K}(\mathsf{T})U)$ and $(V, \beta : V \rightarrow \mathscr{K}(\mathsf{T})V)$, then $f \cdot g \cdot f \eqsim f$ and $g \cdot f \cdot g \eqsim g$.

Properties for Adjoints

The adjoints in a pre-Galois connection uniquely determine each other when the order $\leq$ is a partial order and $\mathscr{K}(\mathrm{T})$ is a faithful functor.

Theorem

If the order $\leq$ is a partial order, and (f, g) and (f, h) are pre-Galois connections between two coalgebras $(U, \alpha : U \to \mathscr{K}(\mathrm{T})U)$ and $(V, \beta : V \to \mathscr{K}(\mathrm{T})V)$ where $\mathscr{K}(\mathrm{T})$ is faithful, then $g = h$ (similarly for the dual case).

Corollary

If $\leq$ is a preorder, and (f, g) and (f, h) are pre-Galois connections between two coalgebras $(U, \alpha : U \to \mathscr{K}(\mathrm{T})U)$ and $(V, \beta : V \to \mathscr{K}(\mathrm{T})V)$ where $\mathscr{K}(\mathrm{T})$ is faithful, then $g \eqsim h$ (similarly for the dual case).

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- *f is monic* iff *g is epic* iff *$g \cdot f = \mathrm{id}_U$;*
- *g is monic* iff *f is epic* iff *$f \cdot g = \mathrm{id}_V$.*

Corollary

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- *f (g) is monic $\Rightarrow g \cdot f \eqsim \mathrm{id}_U$ ($f \cdot g \eqsim \mathrm{id}_V$);*
- *f (g) is epic $\Rightarrow f \cdot g \eqsim \mathrm{id}_V$ ($g \cdot f \eqsim \mathrm{id}_U$).*

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Linking Pre-Galois Connection with Refinement

Given a pre-Galois connection $(f : U \to V, g : V \to U)$, we can extract the relation $R_{(f,g)} \subseteq U \times V$ as follows:

$$R_{(f,g)} = \{(u, v) \mid \mathscr{K}(\mathrm{T})f \cdot \alpha(u) \leq_{\mathscr{K}(\mathrm{T})V} \beta(v)\}$$

or equivalently

$$R_{(f,g)} = \{(u, v) \mid \alpha(u) \leq_{\mathscr{K}(\mathrm{T})U} \mathscr{K}(\mathrm{T})g \cdot \beta(v)\}$$

Theorem

The relation $R_{(f,g)}$ is a simulation.

Corollary

If the preorder $\leq$ be equality $=$, then $R_{(f,g)}$ is a bisimulation.

Linking Pre-Galois Connection with Refinement

Given a pre-Galois connection $(f : U \to V, g : V \to U)$, we can extract the relation $R_{(f,g)} \subseteq U \times V$ as follows:

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Conclusions

- The coalgebraic model for state based components is rebuilt in the Kleisli category.
- The refinement theory for generic state-based components is re-examined for coalgebras in the Kleisli category.
- The notion of pre-Galois connection is defined and some properties are proved.

Future work

- Go deeper into the concept itself
  - Existence of the adjoints in a pre-Galois connection
  - Completeness of pre-Galois connection for refinement
- Application of pre-Galois connection in refinement examples

Thank you!