A comparison of coalgebraic and metric semantics of a CCS-like language

Elisabete Freire [1] Luis Monteiro[2]

[1]Departamento de Matemática
Universidade dos Açores
Portugal

[2]Departamento de Informática
Universidade Nova de Lisboa
Portugal

Motivation

- Sets with families of equivalences
  - less general than metric spaces
  - quite appropriate to objects with a notion of step of construction/observation:
    n-equivalent objects are undistinguishable in the first n-steps
  - can be seen as a metric space with $d(s,t) = 2^{-sup\{n:s \equiv_n t\}}$
- Coalgebras
  - good description of dynamical systems
  - Supported by a well developed general theory - existence of final coalgebras

The language Lsync

$\mathcal{L}_{syn}$ programs

$(a \in)Act$ $(x \in)PVar$ $(c \in)Sync$ sets

- Statements $(s \in)Stat$
  $s ::= a \mid x \mid (s; s) \mid (s + s) \mid (s \parallel s) \mid s \setminus c$
- Guarded Statements $(g \in)GStat$
  $g ::= a \mid (g; s) \mid (g + g) \mid (g \parallel g) \mid g \setminus c$
- Declarations $(D \in)Decl$
  $Decl = PVar \rightarrow GStat$
- Programs $(\pi \in)\mathcal{L}_{syn}$
  $\mathcal{L}_{syn} = Decl \times Stat$

$\mathcal{L}_{syn}$ programs

$(a \in) Act$ $(x \in) PVar$ $(c \in) Sync$ sets

- Statements $(s \in) Stat$
  $s ::= a \mid x \mid (s; s) \mid (s + s) \mid (s \parallel s) \mid s \setminus c$

- Guarded Statements $(g \in) GStat$
  $g ::= a \mid (g; s) \mid (g + g) \mid (g \parallel g) \mid g \setminus c$

- Declarations $(D \in) Decl$
  $Decl = PVar \rightarrow GStat$

- Programs $(\pi \in) \mathcal{L}_{syn}$
  $\mathcal{L}_{syn} = Decl \times Stat$

Resumptions

$(r \in) Res$

$$r ::= E \mid s$$

Transition system for $\mathcal{L}_{syn}$

$\mathcal{T}_{syn}$=(Decl $\times$ Res, Act, $\longrightarrow$,Spec)

- (Act) $a \xrightarrow{a}_D E$;
- (Rec) $\dfrac{g \xrightarrow{a}_D r}{x \xrightarrow{a}_D r}$ if $D(x) = g$;
- (Seq) $\dfrac{s_1 \xrightarrow{a}_D r_1}{s_1; s_2 \xrightarrow{a}_D r_1; s_2}$;
- (Choice) $\dfrac{s_1 \xrightarrow{a}_D r}{\begin{array}{c} s_1 + s_2 \xrightarrow{a}_D r \\ s_2 + s_1 \xrightarrow{a}_D r \end{array}}$;
- (Par) $\dfrac{s_1 \xrightarrow{a}_D r}{\begin{array}{c} s_1 \parallel s_2 \xrightarrow{a}_D r \parallel s_2 \\ s_2 \parallel s_1 \xrightarrow{a}_D s_2 \parallel r \end{array}}$;
- (Sync)
  $\dfrac{s_1 \xrightarrow{c}_D r_1 \qquad s_2 \xrightarrow{\overline{c}}_D r_2}{s_1 \parallel s_2 \xrightarrow{\tau}_D r_1 \parallel r_2}$;
- (Restr)
  $\dfrac{s \xrightarrow{a}_D r}{s \setminus c \xrightarrow{a}_D r \setminus c} \quad a \neq c, \overline{c}.$

Sets with families of equivalence

Advantages

- definitions and proofs are by induction (instead of metric techniques)
- less quantifiers
- it is easier to deal with powerdomains (non determinism)

Set with a family of equivalences (sfe)

- S a set
- $(\equiv_n)_{n\geq 0}$ family of equivalence relations on S such that
  - $\equiv_0$ is the relation $S \times S$;
  - $\equiv_{n+1} \subseteq \equiv_n$ for all $n \geq 0$.

Sets with families of equivalence

Example

For $\{a, b, c\}^*$ we have

$$abcabc \equiv_3 abccbc$$
$$abcabc \equiv_2 ab$$

- Any set S can be viewed as an sfe
  - $\equiv_n = \Delta_S$, $n \geq 1$ (identity relation on S) (*discrete* sfe)

A transition system can be seen as an sfe in in a very simple way

- $s \equiv_n t$ if s and t have the same traces of length n or less

(another way is by using bissimulations)

Set with a family of equivalences

S sfe

- $\equiv_\omega$ denotes the intersection of the $\equiv_n$
- S is *separated* if $\equiv_\omega = \Delta_S$, (identity relation on S)
- Let $(s_n)_{n \geq 0}$ be a sequence in S (abbreviated to $(s_n)_n$ or just (s_n)).
  - (s_n) *converges* to $s \in S$ if $s_n \equiv_n s$ for all n.
    (in a separated sfe limits are unique when they exist.)
  - (s_n) is a *Cauchy* sequence if $s_n \equiv_n s_{n+1}$ for all n.
- S is *complete* if if every Cauchy sequence (s_n) is convergent

Set with a family of equivalences

S, T sfe's

- $f : S \to T$ is *nonexpansive* (resp., *contractive*) if $s \equiv_n t$ implies $f(s) \equiv_n f(t)$ (resp., $f(s) \equiv_{n+1} f(t)$) for all $s, t \in S$ and $n \geq 0$.
- Sfe - category of sfe's and nonexpansive functions
- CSfe - full subcategory of separated and complete sfe's (csfe's)

Banach fixed-point theorem

A contractive function $f : S \to S$ on a complete and separated sfe S has a unique fixed point.

Operational semantics

Domain

$$Decl \times Stat$$

Codomain

Solution of

$$\mathbb{P}_{\mathcal{O}} \cong \{p_\varepsilon\} + \mathcal{P}_{co}(IAct \times \mathbb{P}^{\circ}_{\mathcal{O}})$$

$\{p_\varepsilon\}$ and *IAct* discrete sfe's.

Operational semantics

Base for the operational semantics

$$\mathcal{O}_0 : Res \longrightarrow \mathbb{P}_{\mathcal{O}}$$

- $\mathcal{O}_0(E) = \overrightarrow{p_\varepsilon} = (\overrightarrow{p_\varepsilon}[n])_{n \geq 0}$
  $\overrightarrow{p_\varepsilon}[0] = \emptyset$
  $\overrightarrow{p_\varepsilon}[n+1] = p_\varepsilon$
- $\mathcal{O}_0(s) = (\mathcal{O}_0(s)[n])_{n \geq 0}$
  $\mathcal{O}_0(s)[0] = \emptyset$
  $\mathcal{O}_0(s)[n+1] = \{(b, \mathcal{O}_0(r)[n]) : s \xrightarrow{b}_D r\}$

Operational semantics

Base for the operational semantics

$$\mathcal{O}_0 : Res \rightarrow \mathbb{P}_{\mathcal{O}}$$

- $\mathcal{O}_0(E) = \vec{p_\varepsilon} = (\vec{p_\varepsilon}[n])_{n \geq 0}$
  $\vec{p_\varepsilon}[0] = \emptyset$
  $\vec{p_\varepsilon}[n+1] = p_\varepsilon$
- $\mathcal{O}_0(s) = (\mathcal{O}_0(s)[n])_{n \geq 0}$
  $\mathcal{O}_0(s)[0] = \emptyset$
  $\mathcal{O}_0(s)[n+1] = \{(b, \mathcal{O}_0(r)[n]) : s \xrightarrow{b}_D r\}$

Operational semantics

$$\mathcal{O} : \mathcal{L}_{syn} \rightarrow \mathbb{P}_{\mathcal{O}}$$

$$\mathcal{O}(s) = \mathcal{O}_0(s)$$

Denotational semantics

$$\mathcal{D} : \mathit{Decl} \times \mathit{Res} \to \mathbb{P}_{\mathcal{D}}$$

Codomain

Solution of

$$\mathbb{P}_{\mathcal{D}} \cong \{p_{\varepsilon}\} + \mathcal{P}_{co}(\mathit{Act} \times \mathbb{P}^{\circ}_{\mathcal{O}})$$

$\mathit{Act} = \mathit{IAct} \cup \mathit{Sync}$.

Auxiliary operators

$$\bigoplus : \mathbb{P} \times \mathbb{P} \longrightarrow \mathbb{P}$$

$$(p \bigoplus q) = (p_n \bigoplus_n q_n)_{n \geq 0}$$

$$\bigoplus_n : \mathbb{P}_n \times \mathbb{P}_n \longrightarrow \mathbb{P}_n$$

$$\emptyset \bigoplus_0 \emptyset = \emptyset$$

$$u \bigoplus_{n+1} v = \begin{cases} v & \text{if} \quad u = p_\varepsilon, \\ u & \text{if} \quad v = p_\varepsilon, \\ u \cup v & \text{if } u \neq p_\varepsilon \text{ and } v \neq p_\varepsilon. \end{cases}$$

Auxiliary operators

$$\oplus : \mathbb{P} \times \mathbb{P} \longrightarrow \mathbb{P}$$

$$(p \oplus q) = (p_n \oplus_n q_n)_{n \geq 0}$$

$$\oplus_n : \mathbb{P}_n \times \mathbb{P}_n \longrightarrow \mathbb{P}_n$$

$$\emptyset \oplus_0 \emptyset = \emptyset$$

$$u \oplus_{n+1} v = \begin{cases} v & \text{if } \; u = p_\varepsilon, \\ u & \text{if } \; v = p_\varepsilon, \\ u \cup v & \text{if } u \neq p_\varepsilon \text{ and } v \neq p_\varepsilon. \end{cases}$$

$\oplus$ is nonexpansive

Auxiliary operators

$$\odot : \mathbb{P} \times \mathbb{P} \longrightarrow \mathbb{P}$$

$$(p \odot q) = (p_n \odot_n q_n)_{n \geq 0}$$

$$\odot_n : \mathbb{P}_n \times \mathbb{P}_n \longrightarrow \mathbb{P}_n$$

$$\emptyset \odot_0 \emptyset = \emptyset$$

$$u \odot_{n+1} v = \begin{cases} v & \text{if} \quad u = p_\varepsilon, \\ \{(a, x \odot_n \beta_n(v)) : (a, x) \in u\} & \text{otherwise.} \end{cases}$$

$\odot$ is nonexpansive

Auxiliary mapping

$\mathcal{D}_1 : Res \longrightarrow Env \longrightarrow \mathbb{P}$

$$\mathcal{D}_1(E)(\rho) = \overrightarrow{p_\varepsilon}$$

$$\mathcal{D}_1(a)(\rho) = \overrightarrow{a}$$

$$\mathcal{D}_1(x)(\rho) = \rho(x)$$

$$\mathcal{D}_1(s_1 + s_2)(\rho) = \mathcal{D}_1(s_1)(\rho) \oplus \mathcal{D}_1(s_2)(\rho)$$

$$\mathcal{D}_1(s_1; s_2)(\rho) = \mathcal{D}_1(s_1)(\rho) \odot \mathcal{D}_1(s_2)(\rho)$$

$$\mathcal{D}_1(s_1 \parallel s_2)(\rho) = \mathcal{D}_1(s_1)(\rho) \circledcirc \mathcal{D}_1(s_2)(\rho)$$

$$\mathcal{D}_1(s \setminus c)(\rho) = \mathcal{D}_1(s)(\rho) \oslash c$$

Equivalence between operational and denotational semantics

$$\mathcal{O} = abs \circ \mathcal{D}$$

Proof: By induction on wgt.

Equivalence between operational and denotational semantics

$$\mathcal{O} = abs \circ \mathcal{D}$$

- Case $r = E$:

$$\begin{aligned} \mathcal{O}_0(E) &= \overrightarrow{p_\varepsilon} \\ &= abs \circ \overrightarrow{p_\varepsilon} \\ &= abs \circ \mathcal{D}_1(E)(\rho_D) \\ &= abs \circ \mathcal{D}_0(E). \end{aligned}$$

Equivalence between operational and denotational semantics

$\mathcal{O} = abs \circ \mathcal{D}$

- Case $r = x$:

$$
\begin{array}{rcll}
\mathcal{O}_0(x) & = & (\mathcal{O}_0(x)[n])_{n\geq 0} & \\
 & = & (\mathcal{O}_0(D(x))[n])_{n\geq 0} & \text{(by definition of } \mathcal{T}_{syn}) \\
 & = & \mathcal{O}_0(D(x)) & \text{(by definition)} \\
 & = & abs \circ \mathcal{D}_0(D(x)) & \text{(by indution hypothesis on wgt)} \\
 & = & abs \circ \mathcal{D}_1(D(x))(\rho_D) & \text{(by definition of } \mathcal{D}_0) \\
 & = & abs \circ \rho_D(x) & \text{(because } \rho_D \text{ is a fixed point of } \Psi_D) \\
 & = & abs \circ \mathcal{D}_1(x)(\rho_D) & \text{(by definition of } \mathcal{D}_1) \\
 & = & abs \circ \mathcal{D}_0(x) & \text{(by definition of } \mathcal{D}_0)
\end{array}
$$

Semantics using coalgebras

Advantages

- we deal with the category of sets
- definitions and proofs are by coinduction (instead of fixed points)

The steps

1. Define the appropriate category of coalgebras
2. Define the final coalgebra
3. Define the coalgebra
   - associated to the global system (Operational Semantics)
   - associated with each one of the semantic operations, the syntactical operations are given by composition (Denotational Semantics)

Semantics using coalgebras

The final coalgebra

- $(\mathbb{T}, \iota)$ $\mathbb{T}$ is finitely branching
- $\iota : \mathbb{T} \to \{p_\varepsilon\} + \mathcal{P}_{fin}(A \times \mathbb{T})$

Domain for the operational semantics

- $\mathbb{T}_\mathcal{O} \cong \{p_\varepsilon\} + \mathcal{P}_{fin}(IAct \times \mathbb{T}_\mathcal{O})$
- $\iota_\mathcal{O} : \mathbb{T}_\mathcal{O} \to \{p_\varepsilon\} + \mathcal{P}_{fin}(IAct \times \mathbb{T}_\mathcal{O})$
- $(\mathbb{T}_\mathcal{O}, \iota_\mathcal{O})$ final coalgebra on Set

Operational semantics

$$\mathcal{O}[\![\,.\,]\!] : \mathcal{L}_{syn} \to \mathbb{T}_\mathcal{O}$$

$$\mathcal{O}[\![\, s \,]\!] = \mathcal{O}_0(s)$$

Denotational Semantics

Models and operations

For each operation on the processes of $\mathcal{L}_{syn}$ we define a corresponding operation over the systems $M = (Q, \phi)$

- $\phi : Q \rightarrow \{p_\varepsilon\} + \mathcal{P}_{fin}(Act \times Q)$;

Models and operations

Parallel composition

- $M = (Q, \phi)$, $N = (P, \psi)$
- $M \parallel N = (Q \times P, \phi \parallel \psi)$
  - $\phi \parallel \psi$

$$(1) \quad (q,p) \downarrow_{\phi\parallel\psi} \quad \text{if and only if } q \downarrow_{\phi} \ \text{ and } p \downarrow_{\psi}$$

$$(2) \quad \frac{q \xrightarrow{c}_{\phi} q' \qquad\qquad p \xrightarrow{\overline{c}}_{\psi} p'}{(q,p) \xrightarrow{\tau}_{\phi\parallel\psi} (q',p')}$$

$$(3) \quad \frac{q \xrightarrow{a}_{\phi} q'}{(q,p) \xrightarrow{a}_{\phi\parallel\psi} (q',p)}$$

$$(4) \quad \frac{p \xrightarrow{a}_{\psi} p'}{(q,p) \xrightarrow{a}_{\phi\parallel\psi} (q,p')}.$$

Models and operations

Restriction

- $M = (Q, \phi)$, $c \in Sync$
- $M \setminus c = (Q, \phi \setminus c)$
  - $\phi \setminus c$

$$(1) \quad p \downarrow_{\phi \setminus c} \quad \text{if and only if} \quad p \downarrow_{\phi}$$

$$(2) \quad \frac{p \xrightarrow{a}_{\phi} q, \quad a \notin \{c, \overline{c}\}}{p \xrightarrow{a}_{\phi \setminus c} q}$$

Conclusions

Comparison between the two techniques

- $\mathcal{O}_{\text{sfe}} = \mathcal{O}_{coal}$
- $abs_{\text{sfe}} \circ \mathcal{D}_{\text{sfe}} = abs_{coal} \circ \mathcal{D}_{coal}$
- both operational semantics are obtained directly from the transition system without being necessary to appeal to fixed point results.
- In the coalgebraic definition of the semantics it was interesting to see how semantic operators can be defined with coalgebras instead of using of fixed point techniques.
- The domains used in the coalgebraic approach are more intuitive because they use the finite powerset ($\mathcal{P}_{fin}$) instead of the compact powerset ($\mathcal{P}_{co}$) used in the approach with sfe's

Conclusions

- The definitions based on sfe's are very similar to the ones of the metric techniques but with sfe's we prevent some fixed point constructions and the proofs are almost all by induction which simplifies most of the process.
- We did not make a pure coalgebric approach for the denotational semantics, we introduced a metric technique for the calculation of the environment ρ_D associated with the declaration D, even so we are convinced it would be possible to make a pure approach. It is a subject that we hope to explore in the future.

Auxiliary operators

$$ⓘ : \mathbb{P} \times \mathbb{P} \longrightarrow \mathbb{P}$$

$$(p \textcircled{\shortparallel} q) = (p_n \textcircled{\shortparallel}_n q_n)_{n \geq 0}$$

$$\textcircled{\shortparallel}_n : \mathbb{P}_n \times \mathbb{P}_n \to \mathbb{P}_n$$

$$\emptyset \textcircled{\shortparallel}_0 \emptyset = \emptyset$$

$$u \textcircled{\shortparallel}_{n+1} v = u \| \lfloor_{n+1} v \ \cup \ v \| \lfloor_{n+1} u \ \cup \ u \mid_{n+1} v$$

$$u \| \lfloor_{n+1} v = \begin{cases} v & \text{if} \quad u = p_\varepsilon, \\ \{(a, x \textcircled{\shortparallel}_n \beta_n(v)) : (a, x) \in u\} & \text{otherwise;} \end{cases}$$

$$u \mid_{n+1} v = \cup \{(\tau, x \textcircled{\shortparallel}_n y) : (c, x) \in u, (\overline{c}, y) \in v\}.$$

$\textcircled{\shortparallel}$ is conservative

Auxiliary operators

$$\oslash : \mathbb{P} \times Sync \longrightarrow \mathbb{P}$$

$$(p \oslash c) = (p_n \oslash_n c)_{n \geq 0}$$

$$\oslash_n : \mathbb{P}_n \times Sync \longrightarrow \mathbb{P}_n$$

$$\emptyset \oslash_0 c = \emptyset$$

$$u \oslash_{n+1} c = \begin{cases} p_\varepsilon & \text{if } \quad u = p_\varepsilon, \\ \{(a, x \oslash_n c) : (a, x) \in u, a \neq c, a \neq \overline{c}\} & \text{otherwise.} \end{cases}$$

$\oslash$ is nonexpansive

Action

- $M_a = (\{*, \nabla\}, \phi_a)$
  - ϕ_a

$$(1) \quad * \xrightarrow{a}_{\phi_a} \nabla$$

$$(2) \quad \nabla \downarrow_{\phi_a}$$

Sum

- $M = (Q, \phi)$, $N = (P, \psi)$
- $M + N = ((Q + \{\nabla\}) \times (P + \{\nabla\}), \phi + \psi)$
  - $\phi + \psi$

$$(1) \quad (\nabla, p) \downarrow_{\phi+\psi} \quad \text{if and only if } p \downarrow_{\psi}$$

$$(2) \quad (q, \nabla) \downarrow_{\phi+\psi} \quad \text{if and only if } q \downarrow_{\phi}$$

$$(3) \quad (q, p) \downarrow_{\phi+\psi} \quad \text{if and only if } q \downarrow_{\phi} \text{ and } p \downarrow_{\psi}$$

$$(4) \quad \frac{q \xrightarrow{a}_{\phi} q'}{(q, p) \xrightarrow{a}_{\phi+\psi} (q', \nabla)}$$

$$(5) \quad \frac{p \xrightarrow{a}_{\psi} p'}{(q, p) \xrightarrow{a}_{\phi+\psi} (\nabla, p')}$$

Sequential composition

- $M = (Q, \phi)$, $N = (P, \psi)$
- $M; N = (Q \times P, \phi; \psi)$
  - $\phi; \psi$

$$(1) \quad (q, p) \downarrow_{\phi;\psi} \text{ if and only if } q \downarrow_{\phi} \text{ and } p \downarrow_{\psi}$$

$$(2) \quad \frac{p \xrightarrow{a}_{\psi} p'}{(q, p) \xrightarrow{a}_{\phi;\psi} (q, p')} \text{ if } q \downarrow_{\phi}$$

$$(3) \quad \frac{q \xrightarrow{a}_{\phi} q'}{(q, p) \xrightarrow{a}_{\phi;\psi} (q', p)}$$

Operational semantics - Example 1

$\mathcal{O}((b_1; c) \parallel b_2) = (\mathcal{O}_0((b_1; c) \parallel b_2)[n])_{n \geq 0}$, with

$$\mathcal{O}_0((b_1; c) \parallel b_2)[0] = \emptyset$$

$$\begin{aligned} \mathcal{O}_0((b_1; c) \parallel b_2)[1] &= \{(b_1, \mathcal{O}_0(c \parallel b_2)[0]), (b_2, \mathcal{O}_0(b_1; c)[0]))\} \\ &= \{(b_1, \emptyset), (b_2, \emptyset)\} \end{aligned}$$

$$\begin{aligned} \mathcal{O}_0((b_1; c) \parallel b_2)[2] &= \{(b_1, \mathcal{O}_0(c \parallel b_2)[1]), (b_2, \mathcal{O}_0(b_1; c)[1]))\} \\ &= \{(b_1, \{(b_2, \mathcal{O}_0(c)[0])\}), (b_2, \{(b_1, \mathcal{O}_0(c)[0])\})\} \\ &= \{(b_1, \{(b_2, \emptyset)\}), (b_2, \{(b_1, \emptyset)\})\} \end{aligned}$$

Operational semantics - Example 1

$\mathcal{O}((b_1; c) \parallel b_2) = (\mathcal{O}_0((b_1; c) \parallel b_2)[n])_{n \geq 0}$, with

$$
\begin{aligned}
\mathcal{O}_0((b_1; c) \parallel b_2)[3] &= \{(b_1, \mathcal{O}_0(c \parallel b_2)[2]), (b_2, \mathcal{O}_0(b_1; c)[2])\} \\
&= \{(b_1, \{(b_2, \mathcal{O}_0(c)[1])\}), (b_2, \{(b_1, \mathcal{O}_0(c)[1])\})\} \\
&= \{(b_1, \{(b_2, \emptyset)\}), (b_2, \{(b_1, \emptyset)\})\}
\end{aligned}
$$

$$
\mathcal{O}_0((b_1; c) \parallel b_2)[3 + i] = \{(b_1, \{(b_2, \emptyset)\}), (b_2, \{(b_1, \emptyset)\})\}, i = 1, 2, \ldots
$$

Operational semantics - Example 2

$\mathcal{O}((b_1; c) \parallel (b_2; \overline{c})) = (\mathcal{O}_0((b_1; c) \parallel (b_2; \overline{c}))[n])_{n \geq 0}$, with

$$\mathcal{O}_0((b_1; c) \parallel (b_2; \overline{c}))[0] = \emptyset$$

$$\begin{aligned}\mathcal{O}_0((b_1; c) \parallel (b_2; \overline{c}))[1] &= \{(b_1, \mathcal{O}_0(c \parallel (b_2; \overline{c}))[0]), (b_2, \mathcal{O}_0((b_1; c) \parallel \overline{c})[0])\} \\ &= \{(b_1, \emptyset), (b_2, \emptyset)\}\end{aligned}$$

$$\begin{aligned}\mathcal{O}_0((b_1; c) \parallel (b_2; \overline{c}))[2] &= \{(b_1, \mathcal{O}_0(c \parallel (b_2; \overline{c}))[1]), (b_2, \mathcal{O}_0((b_1; c) \parallel \overline{c})[1])\} \\ &= \{(b_1, \{(b_2, \emptyset)\}), (b_2, \{(b_1, \emptyset)\})\}\end{aligned}$$

Operational semantics - Example 2

$\mathcal{O}((b_1; c) \parallel (b_2; \overline{c})) = (\mathcal{O}_0((b_1; c) \parallel (b_2; \overline{c}))[n])_{n \geq 0}$, with

$$\begin{aligned}
\mathcal{O}_0((b_1; c) \parallel (b_2; \overline{c}))[3] &= \{(b_1, \mathcal{O}_0(c \parallel (b_2; \overline{c}))[2]), (b_2, \mathcal{O}_0((b_1; c) \parallel \overline{c})[2])\} \\
&= \{(b_1, \{(b_2, \mathcal{O}_0(c \parallel \overline{c})[1])\}), (b_2, \{(b_1, \mathcal{O}_0(c \parallel \overline{c})[1])\})\} \\
&= \{(b_1, \{(b_2, \{(\tau, \mathcal{O}_0(E)[0])\}\}), \\
&\qquad (b_2, \{(b_1, \{(\tau, \mathcal{O}_0(E)[0])\})\})\} \\
&= \{(b_1, \{(b_2, \{(\tau, \emptyset)\})\}), (b_2, \{(b_1, \{(\tau, \emptyset)\})\})\}
\end{aligned}$$

$$\begin{aligned}
\mathcal{O}_0((b_1; c) \parallel (b_2; \overline{c}))[4] &= \{(b_1, \mathcal{O}_0(c \parallel (b_2; \overline{c}))[3]), (b_2, \mathcal{O}_0((b_1; c) \parallel \overline{c})[3])\} \\
&= \{(b_1, \{(b_2, \mathcal{O}_0(c \parallel \overline{c})[2])\}), (b_2, \{(b_1, \mathcal{O}_0(c \parallel \overline{c})[2])\})\} \\
&= \{(b_1, \{(b_2, \{(\tau, \mathcal{O}_0(E)[1])\})\}), \\
&\qquad (b_2, \{(b_1, \{(\tau, \mathcal{O}_0(E)[1])\})\})\} \\
&= \{(b_1, \{(b_2, \{(\tau, p_\varepsilon)\})\}), (b_2, \{(b_1, \{(\tau, p_\varepsilon)\})\})\}
\end{aligned}$$

$$\mathcal{O}_0((b_1; c) \parallel (b_2; \overline{c}))[i + 4] = \{(b_1, \{(b_2, \{(\tau, p_\varepsilon)\})\}), (b_2, \{(b_1, \{(\tau, p_\varepsilon)\})\})\} \quad i = 1, 2, \ldots$$