

Completing Fourier's project in mathematical physics

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The memoir of Jean-Baptiste Fourier in 1807 aimed at providing a solution to a specific problem in mathematical physics, the heat equation, and he showed that this particular problem could be solved using classical trigonometric functions. Despite the widespread use of Fourier series in science and engineering it remained unproven for a very long time whether the idea of Fourier to approximate functions with any degree of precision using Fourier series was valid. It was only in 1966 that L. Carleson could prove Luzin's conjecture, namely the point-wise convergence of Fourier series, almost everywhere, except in points of measure zero.

This settled a long-standing problem, but it did not complete Fourier's project of mathematical physics. The original problem was a quadruple of a *BVP*, boundary conditions *BC* (Dirichlet, Neumann, Robin), a given boundary function *BF*, and a domain *D*. The original solution is given in terms of Fourier series (i.e. of trigonometric functions defined on the classical circle). It was given on a circular domain, and for the next two centuries analytical solutions remained possible only for a few domains. A variety of methods were developed, very few analytic, most numerical, but none of these methods could use the classical Fourier projection method.

It was only in 2007, exactly 200 years after Fourier's original memoir that it was shown that solutions of *BVP*'s for any normal polar domain could be expressed in Fourier series. The domains not only included normal polar planar domains, but also 3D domains and Riemann surfaces. Later these solutions were extended even to annular domains and shells (in 2D and 3D). In the stretched co-ordinate system the original domain is transformed into the unit circle or unit sphere and in such a system classical techniques like the separation of variables can be used for solving the transformed equation. Solutions can be found also when the boundaries of the considered domain are defined by more general piecewise continuous functions, and the boundary values described by square integrable, not necessarily continuous, functions.

Here we focus on the Helmholtz equation, which relates to many problems of mathematical physics in electromagnetism, acoustics and vibrations. Regular solutions of this equation are metaharmonic functions and many other problems can be reduced to Helmholtz equation. Here we show solutions of the Helmholtz equation in floral domains.