Decidability and Expressiveness results for Nondeterministic Probabilistic Automata

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joint work with

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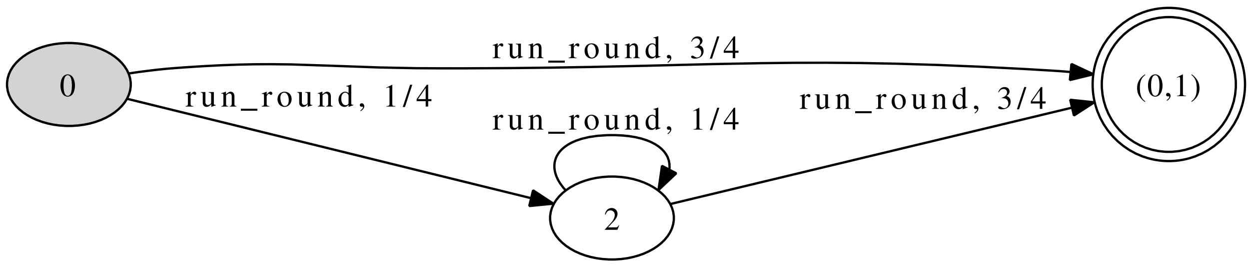
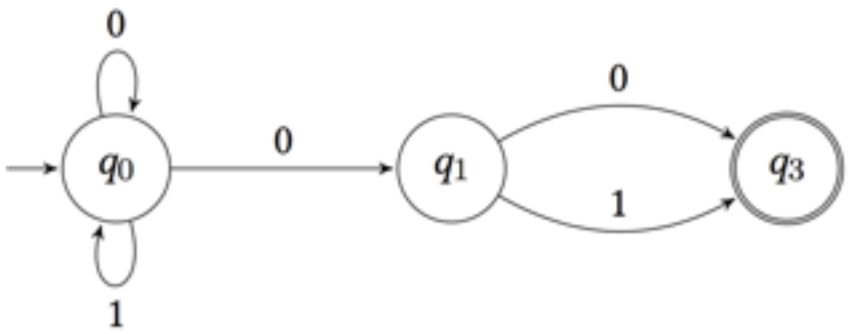
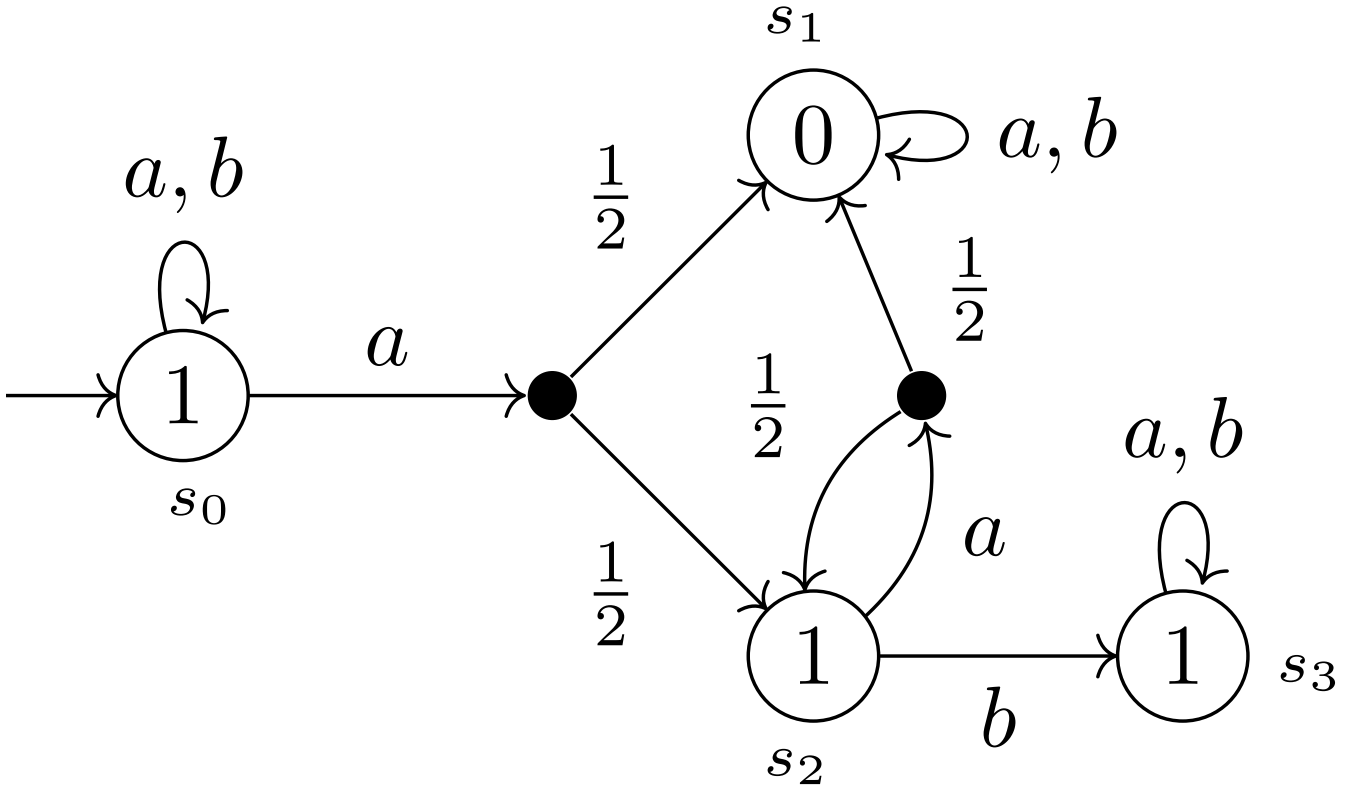
Context

- Probabilistic automata: randomized computation, semantics of programming languages, machine learning.
- Non-deterministic automata: concurrent and distributed systems
- Rabin 80’s: use of nondeterminism and probabilities





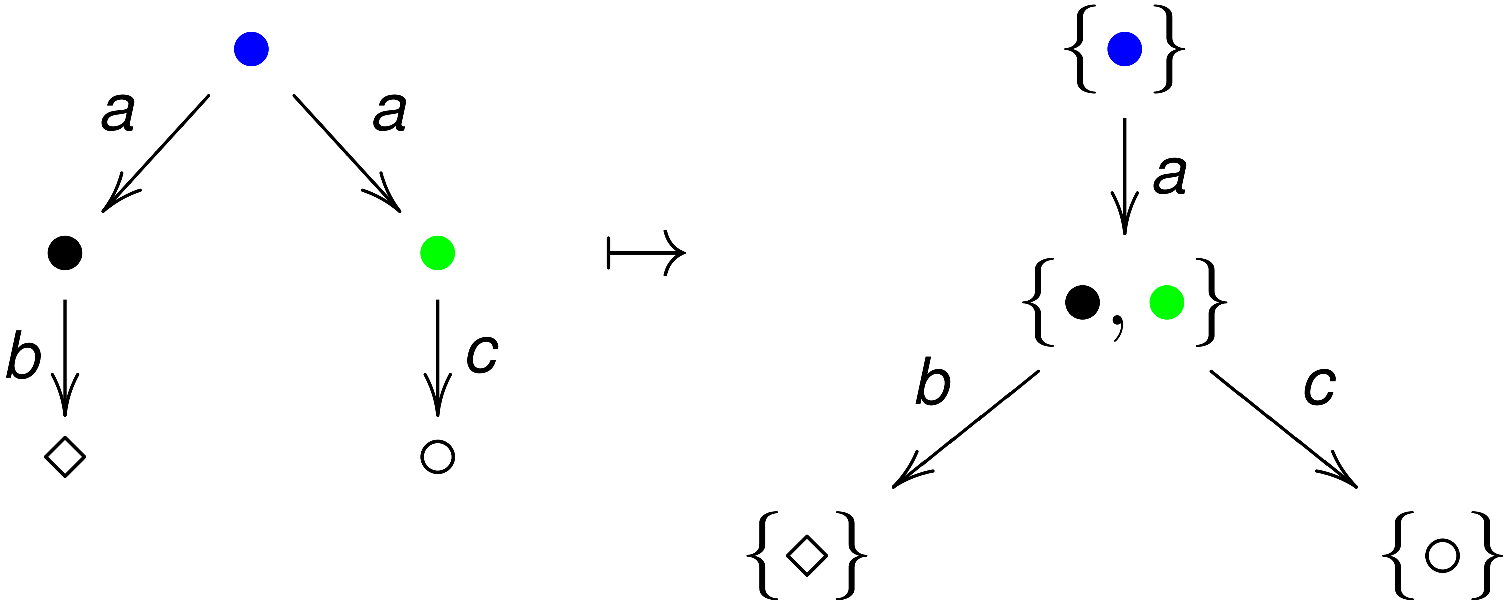
Non-deterministic probabilistic automata



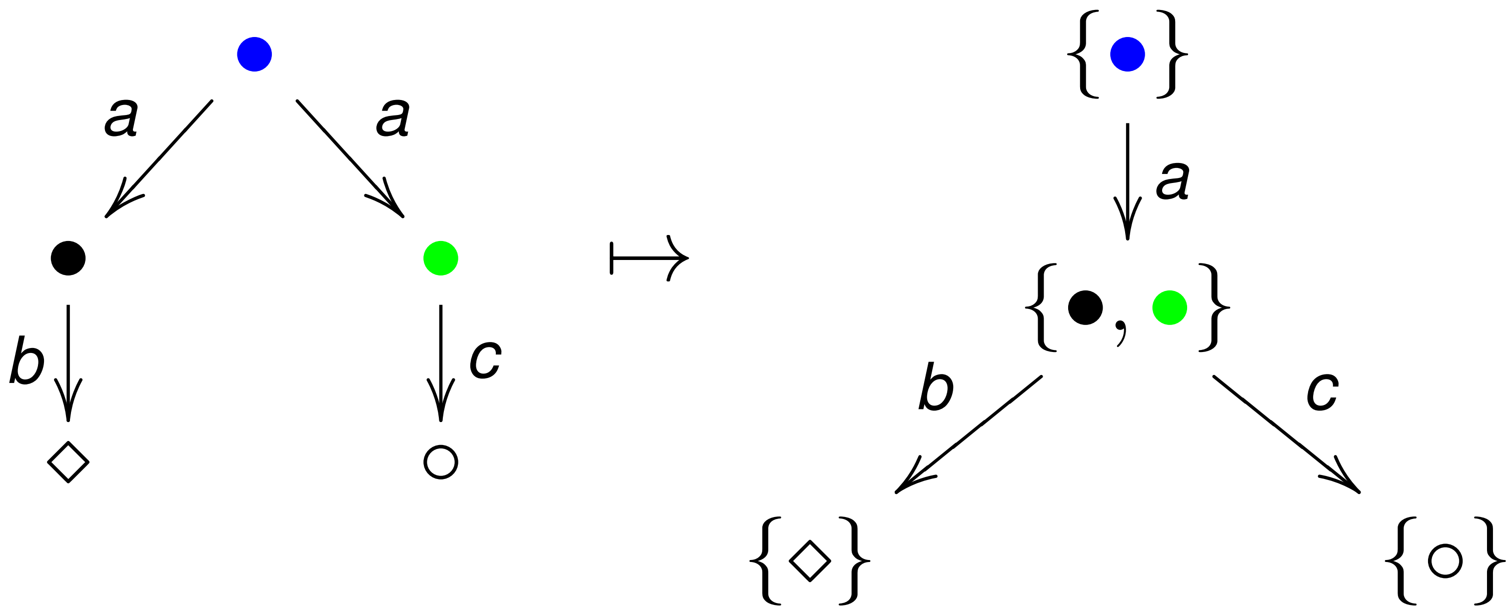
This talk

- Language semantics for NPA
- (Un)decidability
- Expressiveness of non-determinism

Language semantics via powerset construction



Language semantics via powerset construction



$S \to 2 \times \mathcal{P}(S)^A$

$\mathcal{P}(S) \to 2 \times \mathcal{P}(S)^A$

$Q \to 2 \times Q^A$

bisimilarity

language equivalence

Powerset construction

$$\begin{tikzcd}
S \arrow[r, "<o{,}t>"] & 2 \times \mathcal{P}(S)^A \arrow[d] \\
\mathcal{P}(S) \arrow[r, "<\bar{o}{,}\bar{t}>"] & 2 \times \mathcal{P}(S)^A
\end{tikzcd}$$

Powerset construction

$$\begin{tikzcd} S \xrightarrow{<o,t>} 2 \times \mathcal{P}(S)^A \arrow[d] \\ \mathcal{P}(S) \xrightarrow{<\bar{o},\bar{t}>} 2 \times \mathcal{P}(S)^A \end{tikzcd}$$

$$\bar{o}(U) = 1 \iff \exists u \in U . o(u) = 1$$

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$$\bar{o}(U) = \bigvee_{U} o(u)$$

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2 is a JSL

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$$\Big\downarrow$$

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$$\bar{t}(U)(a) = \bigcup_{U} t(u)(a)$$

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JSL and P are closely related in a very general way

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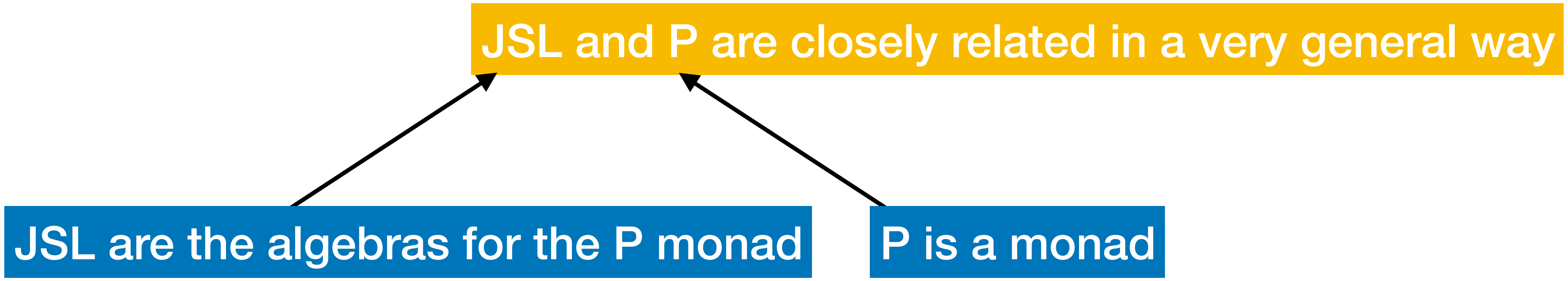

JSL and P are closely related in a very general way

JSL are the algebras for the P monad

P is a monad

Powerset construction

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J

We will used monads and algebras to define a *generalised powerset construction* and have a general *language semantics* for a class of automata

Monads and their algebras

- Monads as *effectful* computation (Moggi 80’s)
- Effects : non-determinism, probabilities, input-output, …

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$$\mu \colon \mathcal{T}\mathcal{T}S \to \mathcal{T}S$$

Monads and their algebras

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- Effects : non-determinism, probabilities, input-output, …

$$\mathcal{T} : Set \to Set \qquad \mathcal{P}(S) = \{U \mid U \subseteq S\}$$

$$\eta \colon S \to \mathcal{T}S \qquad \eta(s) = \{s\}$$

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μ, η **satisfy some reasonable laws e.g.** $\mu \circ \eta = id$

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- Associated with each monad there is a category of (free) algebras

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Definition *An* algebra for a monad (T, η, μ) *is a pair* (X, h) *consisting of a carrier set* X *and a function* $h\colon TX \to X$ *making the following diagrams commute.*

$$\begin{tikzcd}
X \arrow[r, "\eta"] \arrow[dr, equal] & TX \arrow[d, "h"] \\
 & X
\end{tikzcd}
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$\mathcal{P}$ **join-semilattices (with a bottom element)**

Language semantics for more general automata

$$\begin{tikzcd}
S \arrow[r, "{<}o{,}t{>}"] \arrow[d] & O \times \mathcal{T}(S)^A \\
\mathcal{T}(S) \arrow[r, "{<}o{,}t{>}"] & O \times \mathcal{T}(S)^A
\end{tikzcd}$$

$$\bar{o}(s) = \alpha \circ (\mathcal{T} o)(s)$$

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O is a T-algebra

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$$\mu\colon \mathcal{T}\mathcal{T}(S) \to \mathcal{T}(S)$$

T(S) is a T-algebra (free)

Deterministic Probabilistic automata

$$\begin{tikzcd}
S \arrow[r, "<o{,}t>"] \arrow[d] & {[0,1] \times \mathcal{D}(S)^A} \\
\mathcal{D}(S) \arrow[r, "<\bar{o}{,}\bar{t}>"] & {[0,1] \times \mathcal{D}(S)^A}
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$$\mu = m \colon \mathcal{D}\mathcal{D}X \to \mathcal{D}X$$

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m - monad multiplication

Deterministic Probabilistic automata

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Belief automaton

$$\bar{o}(s) = \alpha \circ (\mathcal{T} o)(s)$$

$$\bar{t}(s)(a) = \mu \circ (\mathcal{T} t)(s)(a)$$

$$\alpha = \mathbb{E} \colon \mathcal{D}[0,1] \to [0,1]$$

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$$\mu = \ m \colon \mathcal{D}\mathcal{D}X \to \mathcal{D}X$$

m - monad multiplication

DPA, example

$$\begin{tikzcd}
S \arrow[r, "<o{,}t>"] \arrow[d] & {[0,1] \times \mathcal{D}(S)^A} \\
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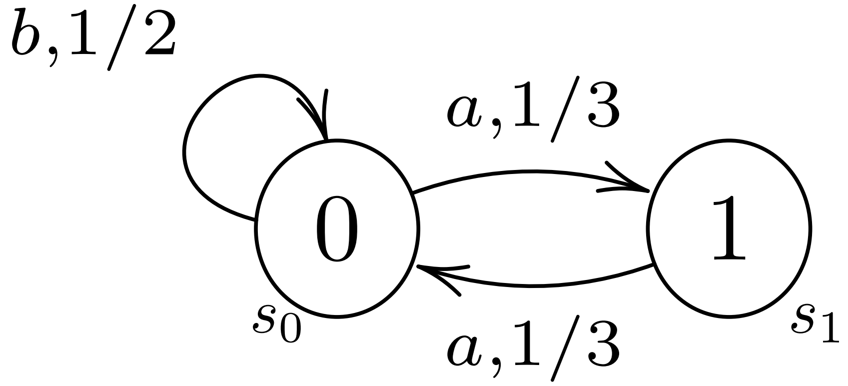

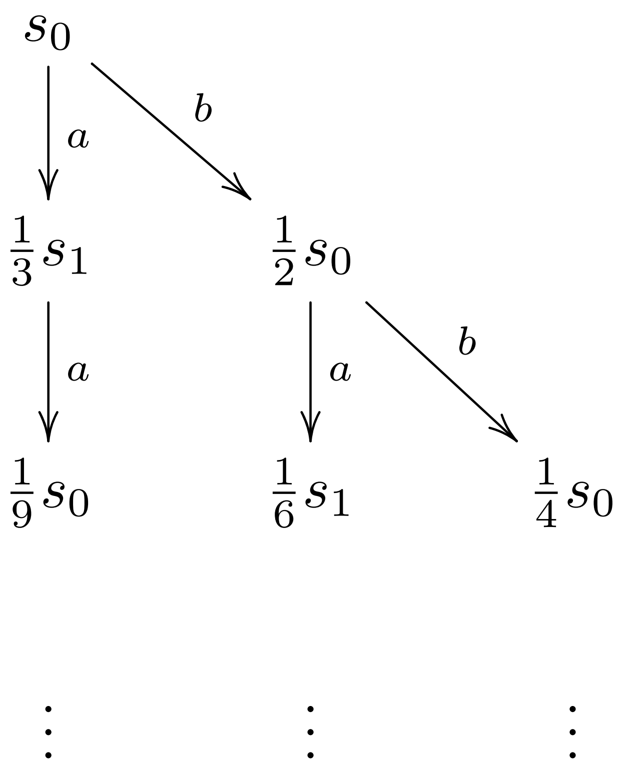

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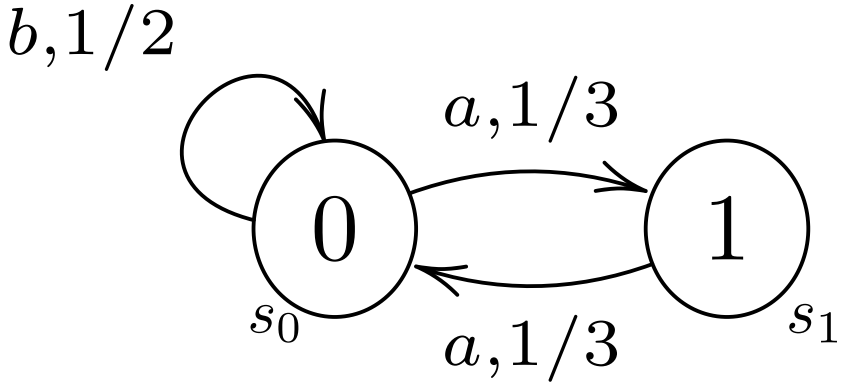
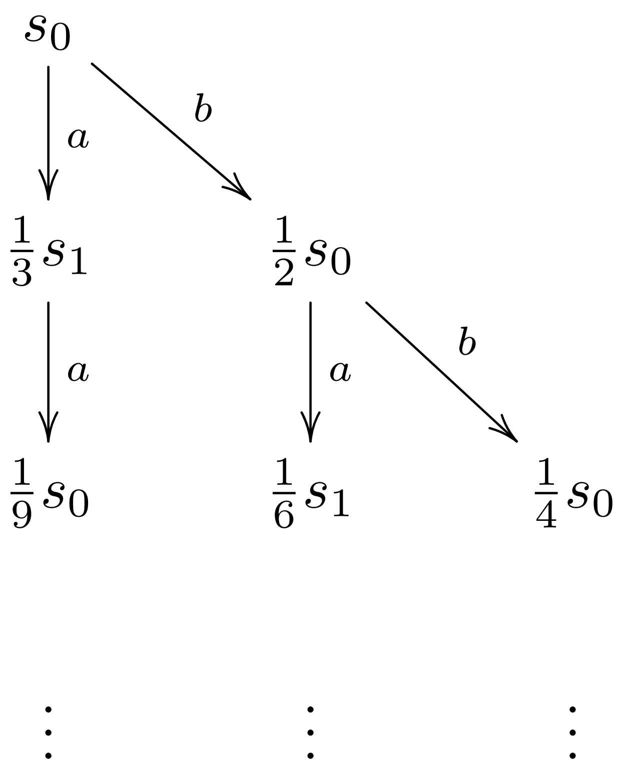


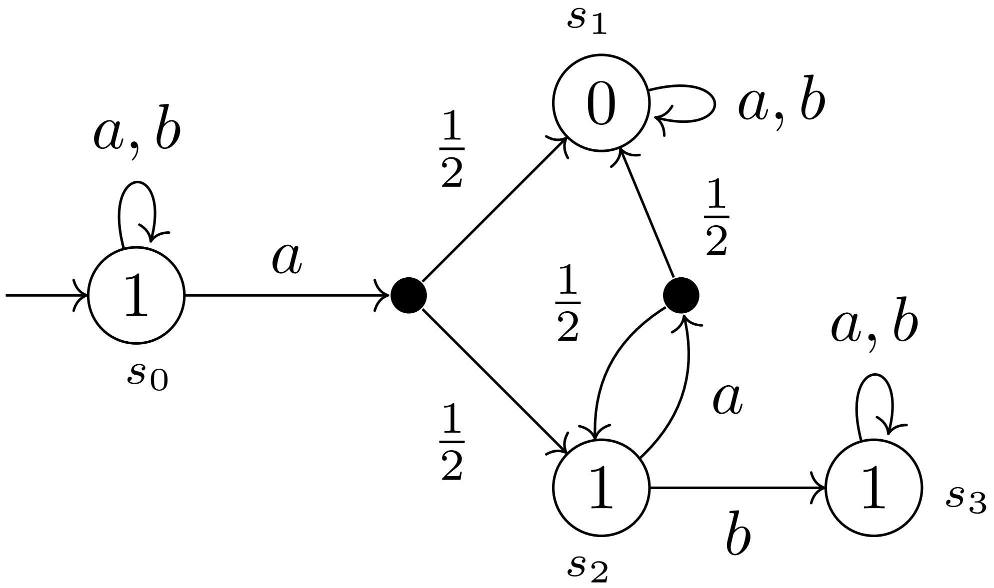
$$\mathcal{L}_s \colon A^* \to [0,1]$$

$$aa \mapsto \frac{1}{9} \times o(s_0) = \frac{1}{9} \times 0 = 0$$

$$ba \mapsto \frac{1}{6} \times o(s_1) = \frac{1}{6} \times 1 = \frac{1}{6}$$

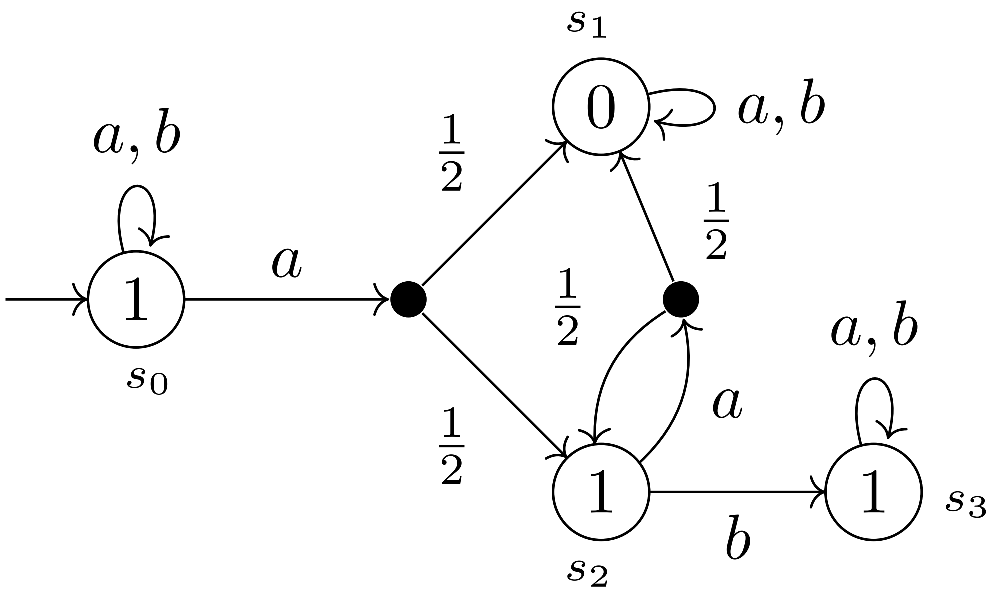


NPA



Definition *A* nondeterministic probabilistic automaton *(NPA) over a (finite) alphabet A is defined by a tuple $(S, s_0, \gamma, \{\tau_a\}_{a \in A})$, where S is a finite set of* states*, $s_0 \in S$ is the* initial state*, $\gamma \colon S \to [0, 1]$ is the* output function*, and $\tau_a \colon S \to \mathcal{P}_c\mathcal{D}S$ are the* transition functions *indexed by inputs $a \in A$.*

NPA



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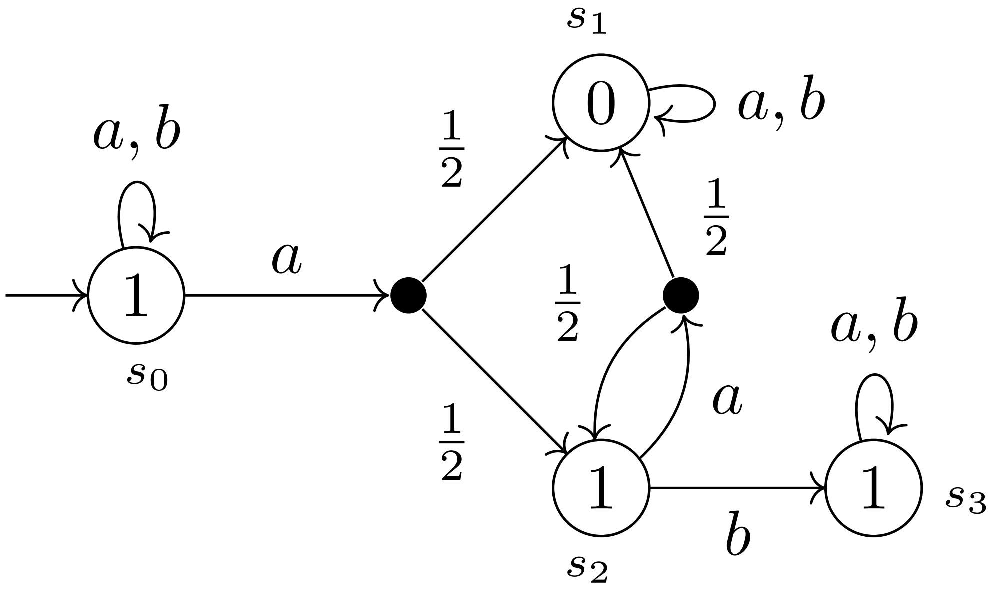
NPA



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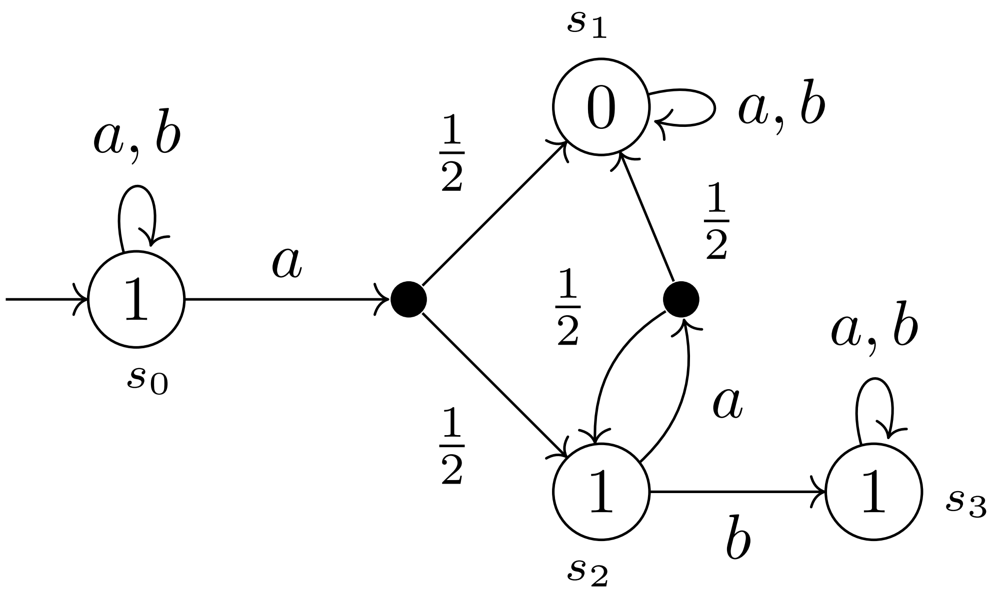
NPA



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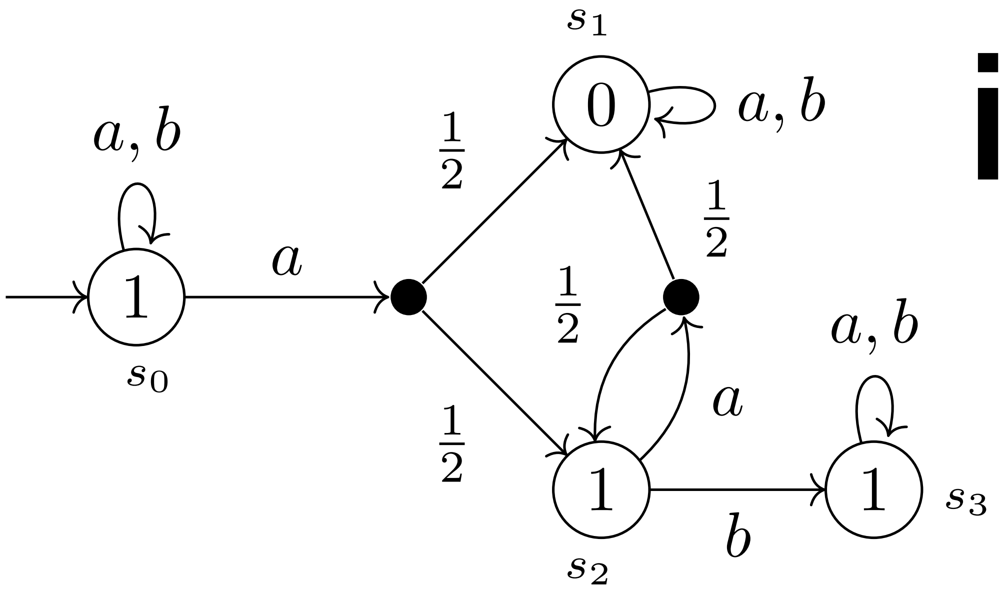
algebra **monad**

Language semantics intuitively



$$\begin{tikzcd}
s_0 \arrow[d, "a"] \\
s_0 \arrow[d, "a"] \\
s_0 \arrow[d, squiggly, no head] \\
1
\end{tikzcd}$$

$$\begin{tikzcd}
s_0 \arrow[d, "a"] \\
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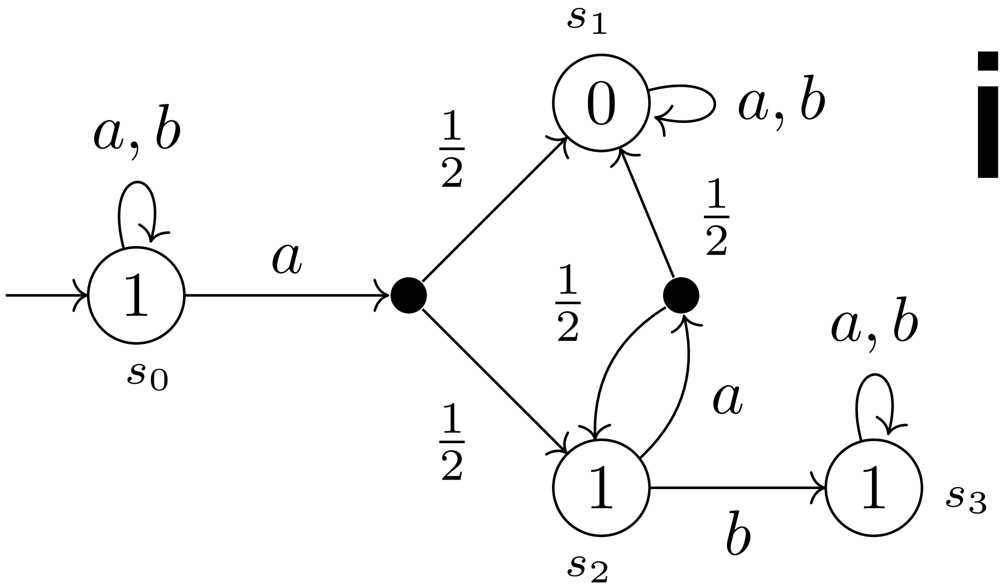
\frac{1}{2}s_1 + \frac{1}{2}s_2 \arrow[d, squiggly, no head] \\
\frac{1}{2}
\end{tikzcd}$$

$$\begin{tikzcd}
s_0 \arrow[d, "a"] \\
\frac{1}{2}s_1 + \frac{1}{2}s_2 \arrow[d, "a"] \\
\frac{1}{2}\left(\frac{1}{2}s_1 + \frac{1}{2}s_2\right) + \frac{1}{2}s_1 \\
= \frac{3}{4}s_1 + \frac{1}{4}s_2 \arrow[d, squiggly, no head] \\
\frac{1}{4}
\end{tikzcd}$$

Language semantics intuitively



$$\begin{tikzcd}
s_0 \arrow[d, "a"] \\
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s_0 \arrow[d, squiggly] \\
1
\end{tikzcd}$$

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\end{tikzcd}$$

max
min
?

Language semantics formally

$$S \to \underset{\underset{\textbf{algebra}}{\uparrow}}{[0,1]} \times \underset{\underset{\textbf{monad}}{\uparrow}}{\mathcal{P}_c\mathcal{D}}(S)^A$$

$\mathcal{P}_c\mathcal{D}$ **is a composite monad**

Language semantics formally

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The semantics should be conservative set DPA

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The semantics should be conservative set DPA

$$\begin{tikzcd}
\mathcal{D}[0,1] \arrow[d, "\{-\}"'] \arrow[dr, "\mathbb{E}"] & \\
\mathcal{P}_c\mathcal{D}[0,1] \arrow[r, "o"] & {[0,1]}
\end{tikzcd}$$

Language semantics formally

$$\begin{tikzcd}
\mathcal{D}[0,1] \arrow[d, "\{-\}"'] \arrow[dr, "\mathbb{E}"] & \\
\mathcal{P}_c\mathcal{D}[0,1] \arrow[r, "o"] & {[0,1]}
\end{tikzcd}$$

Proposition 1. *Any* $\mathcal{P}_c\mathcal{D}$*-algebra on* $[0,1]$ *extending* $\mathbb{E}\colon \mathcal{D}[0,1] \to [0,1]$ *is of the form* $\mathcal{P}_c\mathcal{D}[0,1] \xrightarrow{\mathcal{P}_c\mathbb{E}} \mathcal{P}_c[0,1] \xrightarrow{\alpha} [0,1]$, *where* α *is a* $\mathcal{P}_c$*-algebra.*

Proposition 2. *The only* $\mathcal{P}_c$*-algebras on the convex set* $[0,1]$ *are* min *and* max.

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Language semantics formally

Proof:

Basic convex algebra facts

$\mathsf{Conv}(\{\{0\},\{1\},[0,1]\}) \;=\; \mathcal{P}_c[0,1]$

$\mathcal{P}_c[0,1] \xrightarrow{\;\alpha\;} [0,1]$

α is completely determined by $\alpha([0,1])$

$\alpha([0,1]) = 0$ or $\alpha([0,1]) = 1$

min or max

$$\begin{tikzcd}
\mathcal{D}[0,1] \arrow[d, "\{-\}"'] \arrow[dr, "\mathbb{E}"] & \\
\mathcal{P}_c\mathcal{D}[0,1] \arrow[r, "o"] & {[0,1]}
\end{tikzcd}$$

on $[0,1]$ *extending* $\mathbb{E}\colon \mathcal{D}[0,1] \to [0,1]$ *is of the form* $\mathcal{P}_c\mathcal{D}[0,1] \xrightarrow{\mathcal{P}_c\mathbb{E}} \mathcal{P}_c[0,1] \xrightarrow{\alpha} [0,1]$, *where* α *is a* $\mathcal{P}_c$*-algebra.*

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This formally explains why in the literature min and max are the only functions used

on $[0,1]$ *extendi*

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Summary, so far

$$\begin{tikzcd} S \to [0,1] \times \mathcal{P}_c\mathcal{D}(S)^A \arrow[d, squiggly, "\text{generalised powerset construction}"] \\ \mathcal{P}_c\mathcal{D}(S) \to [0,1] \times \mathcal{P}_c\mathcal{D}(S)^A \end{tikzcd}$$

min and max are the only algebras

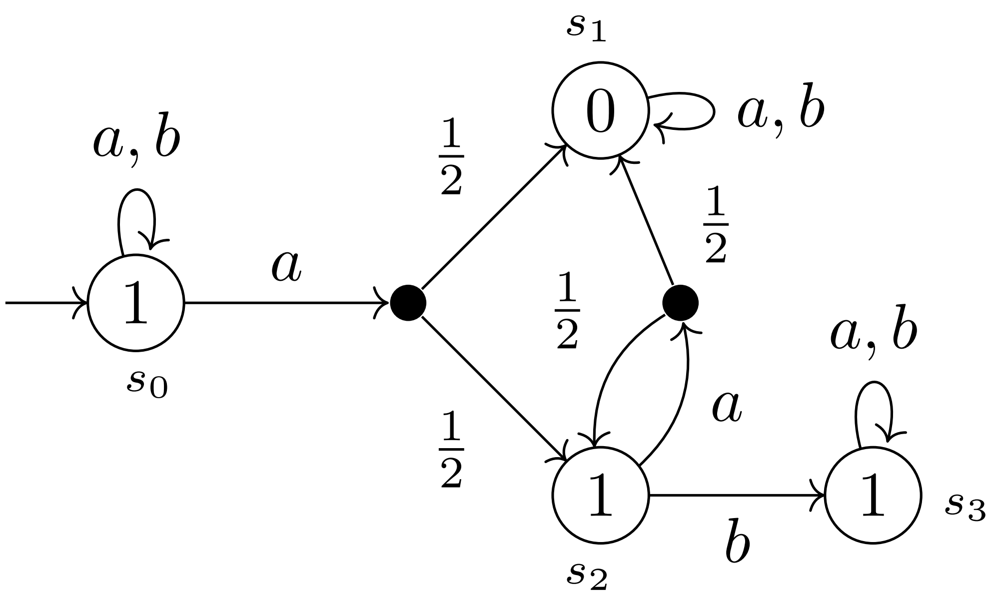
Deterministic automaton

Language semantics as usual using min/max to combine set of output results

Next

- Non-determinism brings a lot of trouble… is it necessary?
- Given two NPAs can we decide whether they are equivalent?

Is non-determinism really important?



$\mathcal{L}_a \colon \{a, b\}^* \to [0, 1]$ by $\mathcal{L}_a(u) = 2^{-n}$

n - length longest sequence of a's in u

Theorem 1. *NPAs are more expressive than DPAs. Specifically, there is no DPA, or even WFA, accepting* $\mathcal{L}_a$.

Is non-determinism really important?

Theorem *There is a language over a unary alphabet that is recognizable by an NPA but not by any WFA (and in particular any DPA).*

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Proof uses results of Linear recurrence sequences

Skolem-Macher-Lech Theorem

Cayley-Hamilton Theorem

(Un)Decidability

Theorem *Equivalence of NPAs is undecidable when* $|A| \geq 2$ *and the* $\mathcal{P}_c\mathcal{D}$-*algebra on* $[0,1]$ *extends the usual* $\mathcal{D}$-*algebra on* $[0,1]$.

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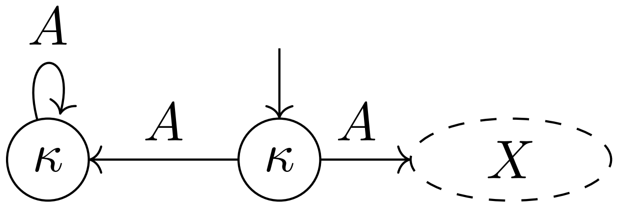

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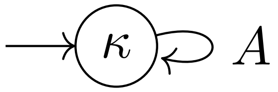

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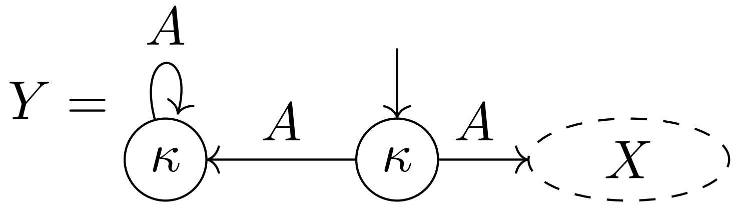

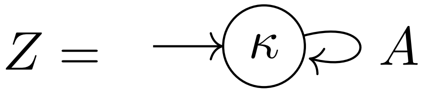

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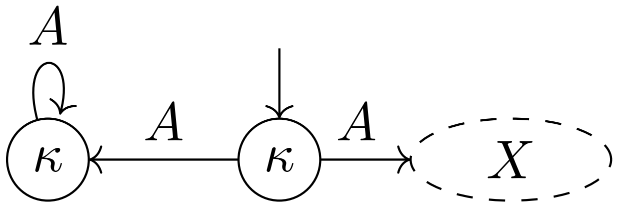

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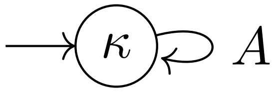

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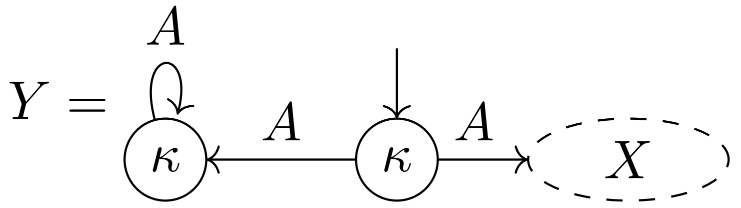

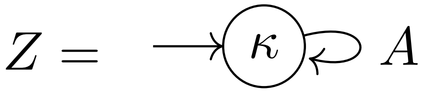

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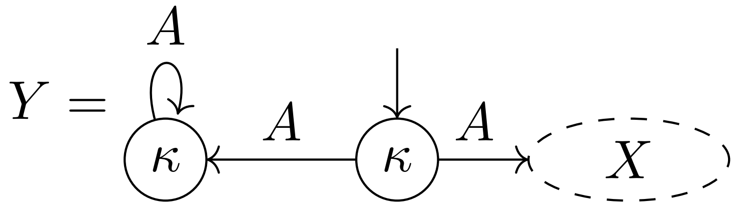

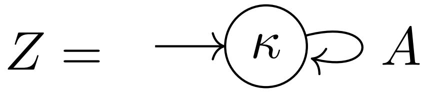

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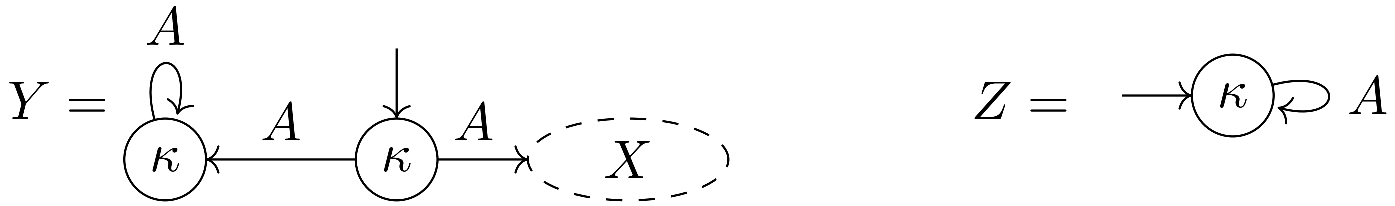

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Threshold problems undecidable for alphabets of size at least 2 [Blondel, Canterini'03]

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(Un)Decidability

- (Un)decidability of threshold problem not know for |A|=1
- Reduction to the *Positivity problem* for LRS
- Decidability of *Positivity* open for >80 years
- Decision procedure would entail breakthroughs in open problems in number theory (algorithm to compute the homogeneous Diophantine approximation type for a class of transcendental numbers)

Corollary 2. *The Positivity problem for linear recurrence sequences can be reduced to the equivalence problem of NPAs over a unary alphabet.*

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Given $c \in [0,1)$ and two weighted languages $l_1, l_2 \colon A^* \to [0,1]$, we define

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- Approximate the metric for any desired non-zero error:

Theorem *There is a procedure that given $c \in [0,1)$, $\kappa > 0$, and computable functions $l_1, l_2 \colon A^* \to [0,1]$ outputs $x \in \mathbb{R}_+$ such that $|d_c(l_1, l_2) - x| \leq \kappa$.*

Conclusions

- Non-determinism + probabilities subtle from a semantics and algorithmic perspective
- Connections between and interplay of convex algebra, number theory, category theory
- Approximation techniques interesting for verification applications - different metrics?

Questions?