



Moessner and the theorem that started it all

Alexandra Silva

Moessner's Construction ($n=4$)

Moessner's Construction (n=4)

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24

Moessner's Construction (n=4)

1 2 3 ~~4~~ 5 6 7 ~~8~~ 9 10 11 ~~12~~ 13 14 15 ~~16~~ 17 18 19 ~~20~~ 21 22 23 ~~24~~

Moessner's Construction (n=4)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
1	3	6		11	17	24		33	43	54		67	81	96		113	131	150		171	193	216	

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1				16				81				256				625				1296			

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1				16				81				256				625				1296			
1 ⁴				2 ⁴				3 ⁴				4 ⁴ .				5 ⁴				6 ⁴			...

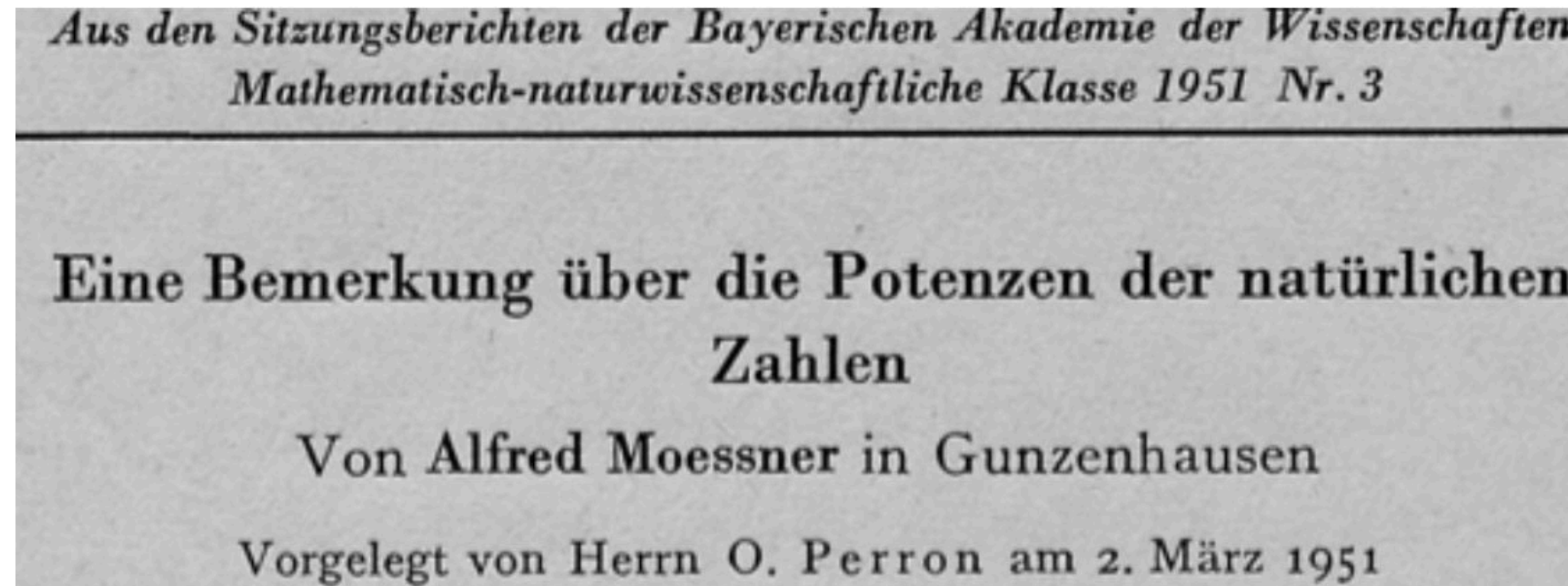
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1	4			15	32			65	108			175	256			369	500			671	864		
1				16				81				256				625				1296			
1^4				2^4				3^4				4^4				5^4				6^4			...

Moessner's Conjecture: This works for all $n \in \mathbb{N}$!

History

1951 – Moessner conjectures it



1952 – Perron proves it

1952 – Paasche and Salie generalize it

1966 – Long presents alternative proof and generalization

2010 – Hinze, Rutten & Niqui present new proofs of just Moessner's theorem

Paasche's Observation

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1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 ...

Paasche's Observation

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Paasche's Observation

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	...
	2		6	11		18	26	35		46	58	71	85		101	118	136	155	175		

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			6			24	50			96	154	225			326	444	580	735			

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										120					720	1764					

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						24				120	274				600	1044	1624				
										120					720	1764					
															720						

1, 2, 6, 24, 120, . . . = 1!, 2!, 3!, 4!, 5!, . . .

What if we increment the increment by one in each step?

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	...
	2	5		10	16	23	31	40		51	63	76	80	95	...					
	2			12	28	51	82			133	196	272	352	...						
				12	40	91				224	420	692	...							
				12	52					276	696	...								
				12						288	984	...								
										288	1272	...								
										288	...									
										288	...									
										288	...									

1, 2, 12, 288, ... ?

What if we increment the increment by one in each step?

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	...
	2	5		10	16	23	31	40		51	63	76	80	95	...					
	2			12	28	51	82			133	196	272	352	...						
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										288	1272	...								
										288	...									
										288	...									
										288	...									

$$1, 2, 12, 288, \dots ? \quad = \quad 1!, 2!1!, 3!2!1!, 4!3!2!1!, \dots = 1!!, 2!!, 3!!, 4!!, \dots$$

Superfactorials!

Yet another Generalization (Long & Salie)

a $a+d$ $a+2d$ $a+3d$ $a+4d$ $a+5d$ $a+6d$ ~~$a+7d$~~ $a+8d$ $a+9d$ $a+10d$ ~~$a+11d$~~

a $a+d$ $a+2d$ ~~$a+3d$~~ $a+4d$ $a+5d$ $a+6d$ ~~$a+7d$~~ $a+8d$ $a+9d$ $a+10d$ ~~$a+11d$~~

a $2a+d$ ~~$3a+3d$~~ $4a+7d$ $5a+12d$ ~~$6a+18d$~~ $7a+26d$ $8a+35d$ $9a+45d$

a ~~$3a+d$~~ $7a+8d$ ~~$12a+20d$~~ $19a+46d$ ~~$27a+81d$~~

a $8a+8d$ $27a+54d$

a $(a+d) \cdot 8$ $(a+2d) \cdot 27$

Yet another Generalization (Long & Salie)

a	a+d	a+2d	a+3d	a+4d	a+5d	a+6d	a+7d	a+8d	a+9d	a+10d	a+11d
a	a+d	a+2d	a+3d	a+4d	a+5d	a+6d	a+7d	a+8d	a+9d	a+10d	a+11d
a	2a+d	3a+3d		4a+7d	5a+12d	6a+18d		7a+26d	8a+35d	9a+45d	
a	3a+d			7a+8d	12a+20d			19a+46d	27a+81d		
a				8a+8d				27a+54d			
a			(a+d) . 8					(a+2d) . 27			

$$a \cdot 1^{n-1}, (a + d) \cdot 2^{n-1}, (a + 2d) \cdot 3^{n-1}, \dots$$

Proofs

Classical proofs: manipulate binomial coefficients

Hinze (2010): calculation scans (functional programming)

Niqui & Rutten: coinductive approach

Not obvious if generalizable

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Dexter's View on Moessner's Theorem

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Change perspective: Do not think of the algorithm as working on streams, but as a **triangle transformation** process!

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1	1	1	1	1		1	1	1	1	1		1	1	1	1	1		1	1	1	1	1
1	2	3	4			5	6	7	8			9	10	11	12			13	14	15	16	
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First triangle: Pascal triangle; all have Pascal property

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1	2	3	4			5	6	7	8			9	10	11	12			13	14	15	16	
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1	2	3	4			5	6	7	8			9	10	11	12			13	14	15	16	
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1	1	1	1	1		1	1	1	1	1		1	1	1	1	1		1	1	1	1	1
1	2	3	4			5	6	7	8			9	10	11	12			13	14	15	16	
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1	1	1	1	1		1	1	1	1	1		1	1	1	1	1		1	1	1	1	1
1	2	3	4			5	6	7	8			9	10	11	12			13	14	15	16	
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Complete the triangle using the Pascal property.

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1	1	1	1	1		1	1	1	1	1		1	1	1	1	1		1	1	1	1	1
1	2	3	4			5	6	7	8			9	10	11	12			13	14	15	16	
1	3	6				11	17	24				33	43	54				67	81	96		
1	4					15	32					65	108					175	256			
1						16						81						256				

One can describe this process as operations on formal power series on two variables, and this yields a proof of Moesser's theorem and its generalizations **all at once!**

Dexter’s View on Moessner’s Theorem

Chan
triang

JOURNAL ARTICLE

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On Moessner’s Theorem

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Dexter Kozen and Alexandra Silva

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The American Mathematical Monthly
Vol. 120, No. 2 (February 2013), pp. 131-139 (9 pages)

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riables,
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The first of many papers!

refine by coauthor
Dexter Kozen (37)
Tobias Kappé (35)

The process

The process



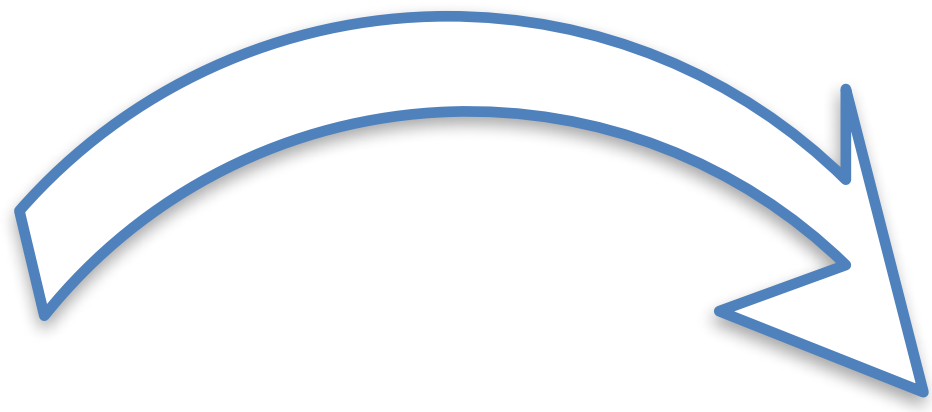
The process



The process



The process



8.30am the next day!

8.30am the next day!



Dexter

Dexter Kozen: A Winning Combination of Brilliance, Depth, and Elegance

David Harel

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David Harel

*Dexter would have read it and would probably have thought:
“Oh yea, that’s kind of interesting, and yes, I can see how
those proofs go; and, by the way, here is how to continue the
work and get far stronger results, and in a much nicer way”.*

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(...)

*A work session with Dexter was always a combination of
exhilaration and frustration.*

Dexter Kozen: A Winning Combination of Brilliance, Depth, and Elegance

David Harel

Dexter Kozen: A Winning Combination of Brilliance, Depth, and Elegance

David Harel

He had the uncanny ability to bring in, out of the blue, unexpected notions from seemingly unrelated branches of mathematics – often from algebra or topology – which either you'd have never heard of, or you'd have long-forgotten. And these then turned out not only to be relevant but to make everything you wanted to do fall into place, and in a concise, exotically elegant and often stunning way.

Dexter

Dexter



2008

Dexter



2008



Kleene Coalgebra
Alexandra Silva

2010

Dexter



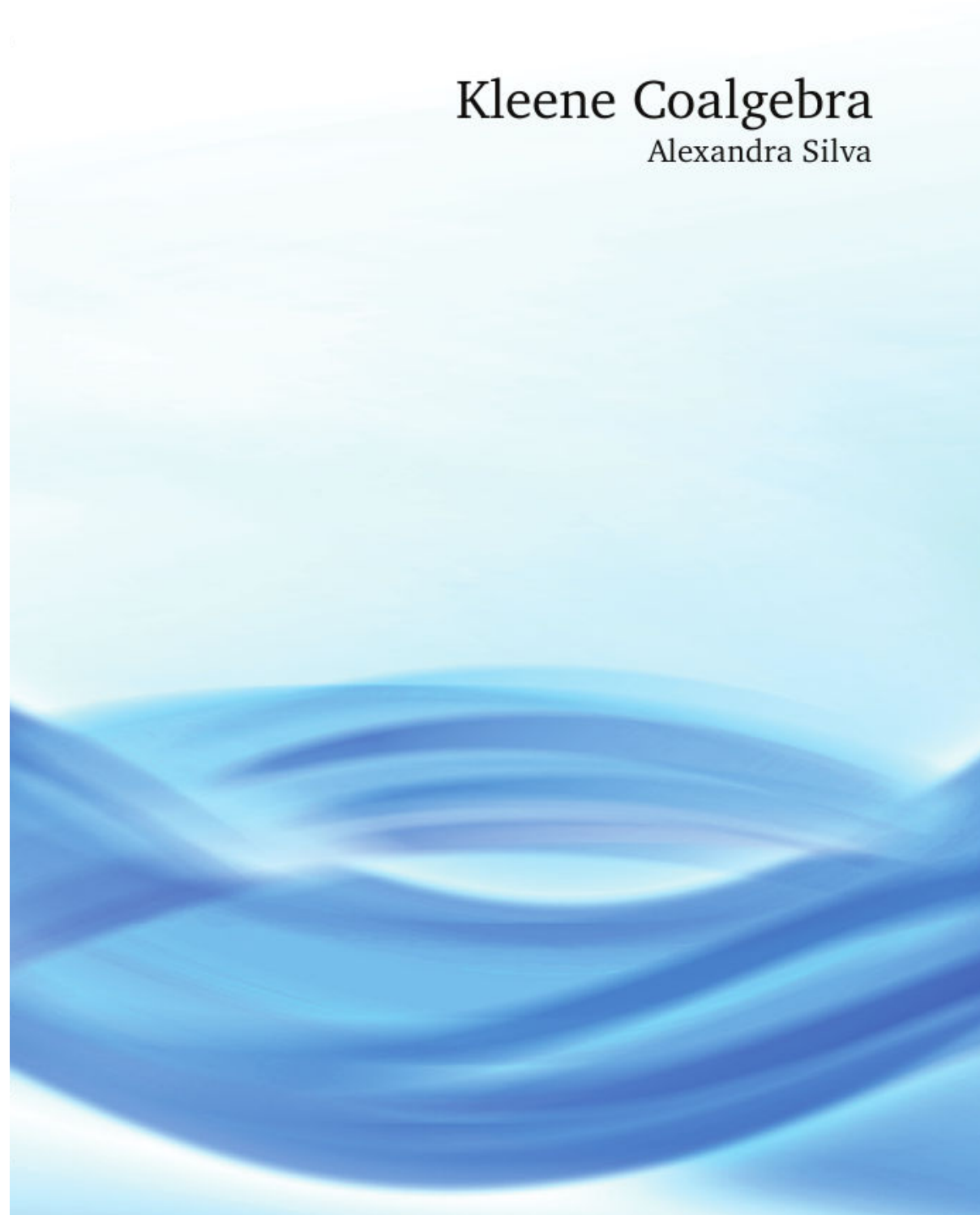
2008



2011

Kleene Coalgebra
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2010



Dexter



2008

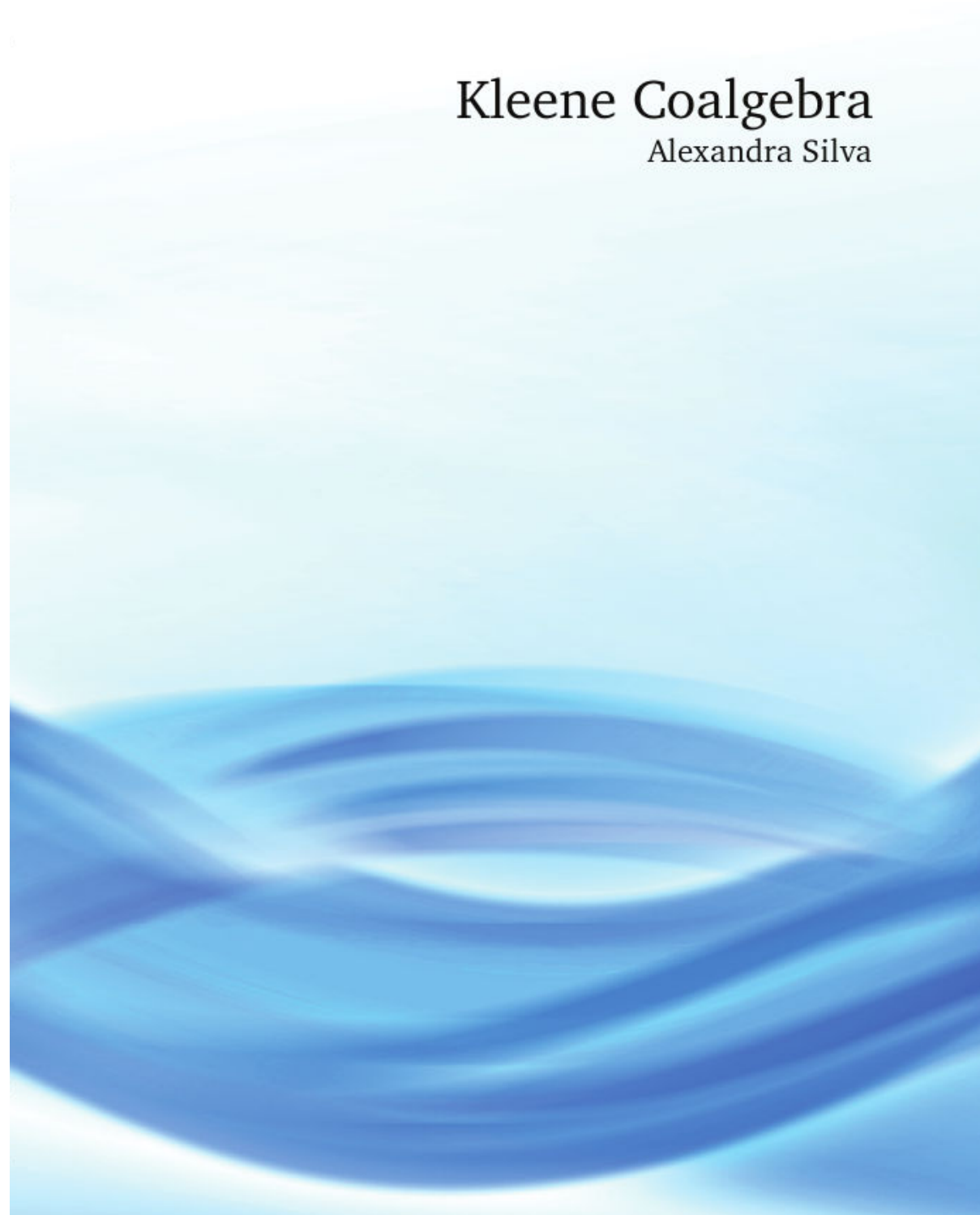


2011



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2010



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2008



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2010



2014



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2015-...



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Carolyn Anderson
Wellesley College



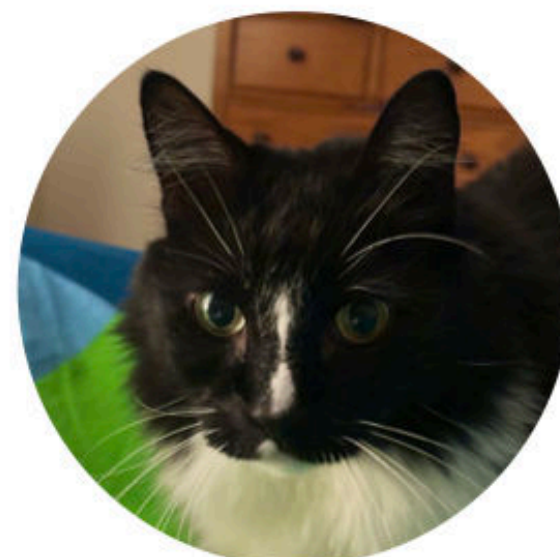
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Carbon
Chez Foster



David Darais
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Arjun Guha
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Jules Jacobs
Cornell



Tobias Kappé
Leiden



Dexter Kozen
Cornell



Mae Milano
Princeton



Mark Moeller
Cornell



Mark Reitblatt
Cornell



Cole Schlesinger
Galois



Alexandra Silva
Cornell



Steffen Smolka
Google



Laure Thompson
Princeton



Jana Wagemaker
Nijmegen



Dave Walker
Princeton

Thanks for
all the fun
times, and
hoping for
many more
to come!

