An algebraic framework to reason about concurrency

Alexandra Silva

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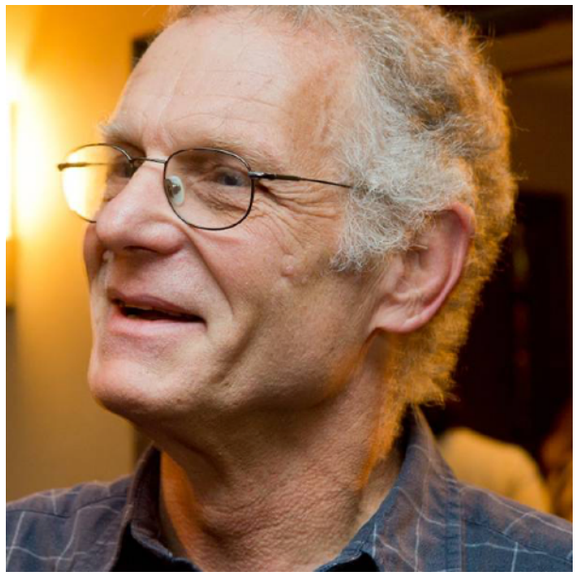
UCL ENGINEERING
Change the world

UCL

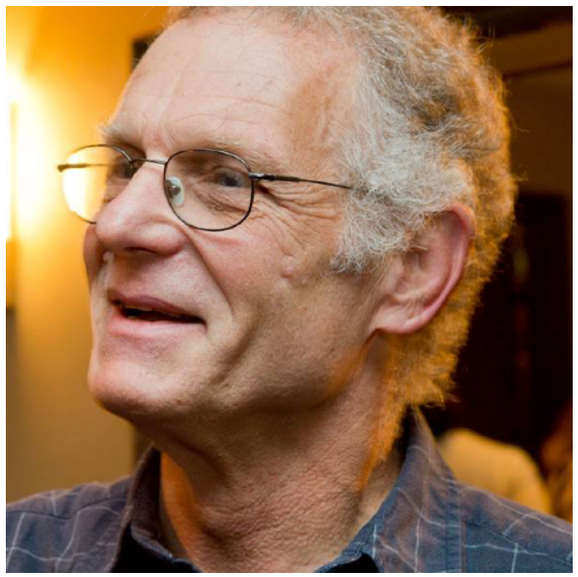
Context

Context



Kleene Algebra as an algebraic foundation for program verification

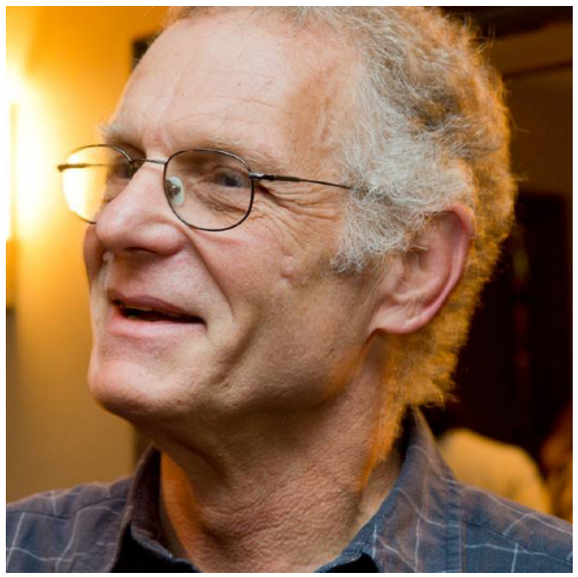
Context



Kleene Algebra as an algebraic foundation for program verification

Completeness

Context



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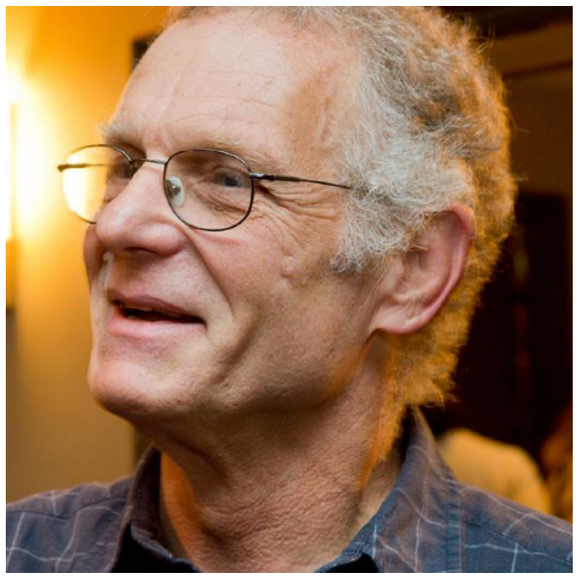

Kleene Algebra as an algebraic foundation for program verification

Completeness

Equivalence

Applications

Program Schematology
Compiler Optimization
and much more!

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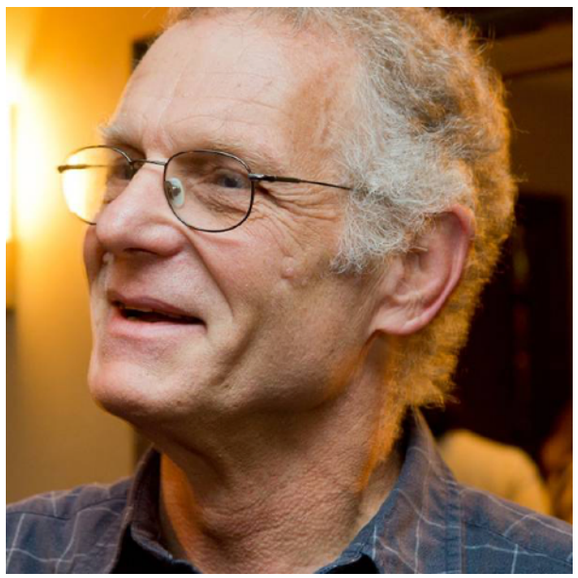

Kleene Algebra as an algebraic foundation for program verification

Completeness

Equivalence

Applications

Program Schematology
Compiler Optimization
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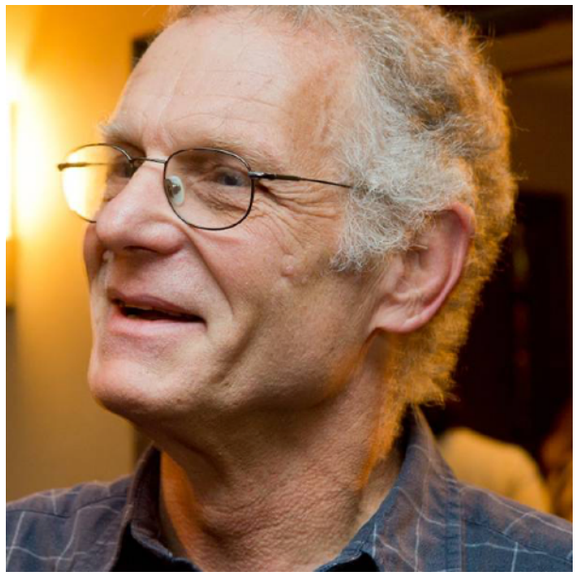
Extensions

KAT, SKAT
KAT+B! ,…

Context



Kleene Algebra as an algebraic foundation for program verification

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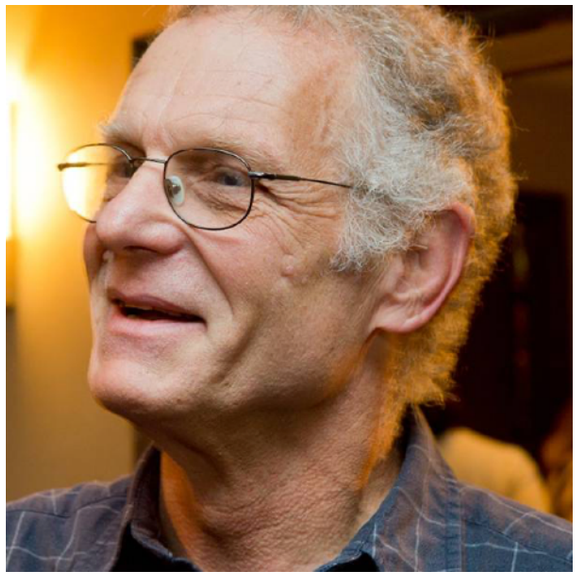

Kleene Algebra as an algebraic foundation for program verification

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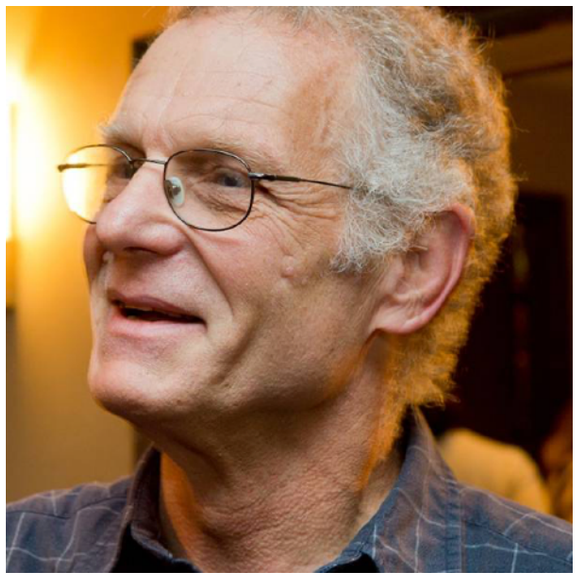


Kleene Algebra as an algebraic foundation for program verification

PROBABILISTIC NETKAT

Kleene Algebra: why?

Kleene Algebra: why?

Compositional Semantics

Kleene Algebra: why?

Compositional Semantics

Scalable verification

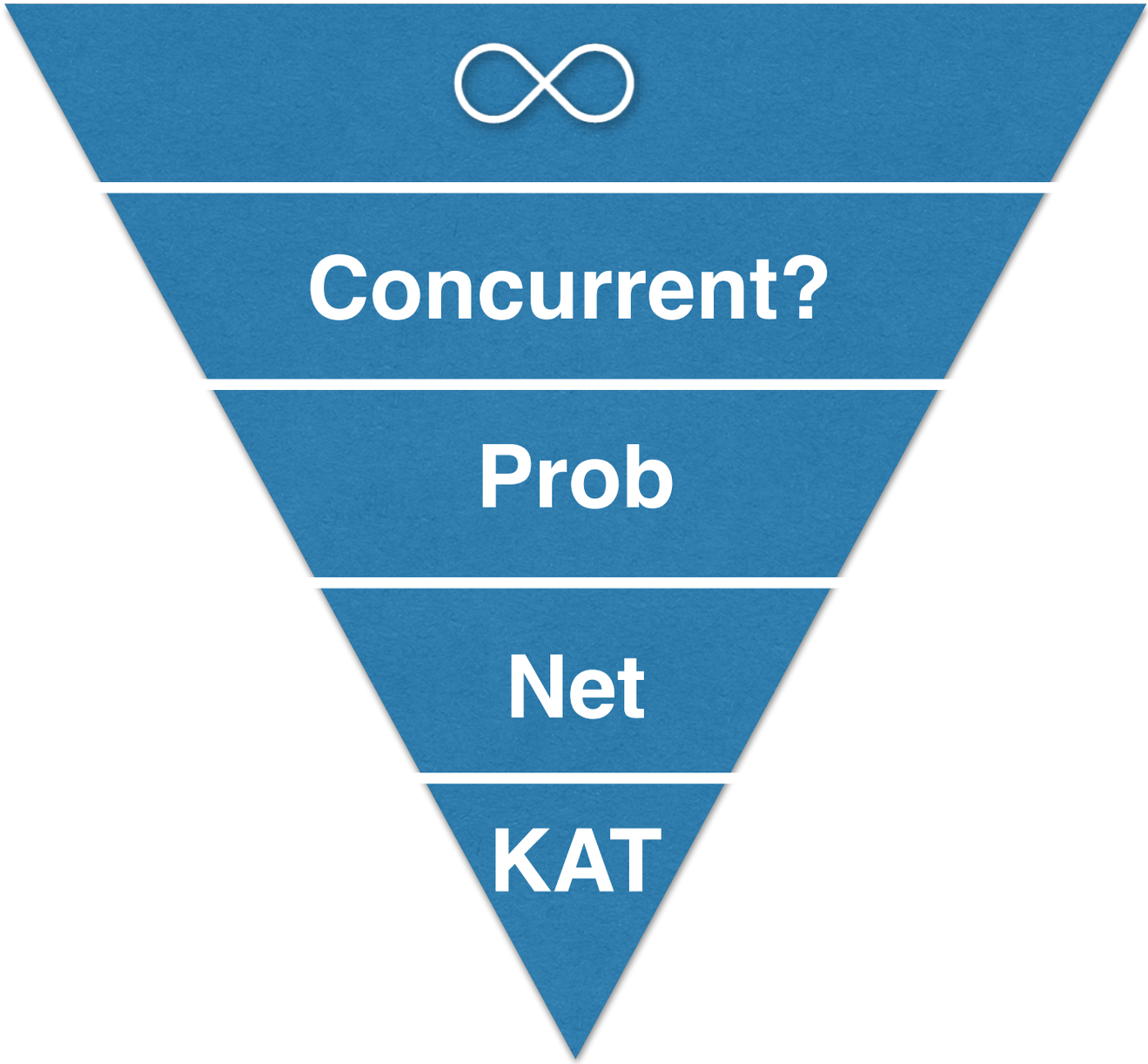
Kleene Algebra: why?

Compositional Semantics

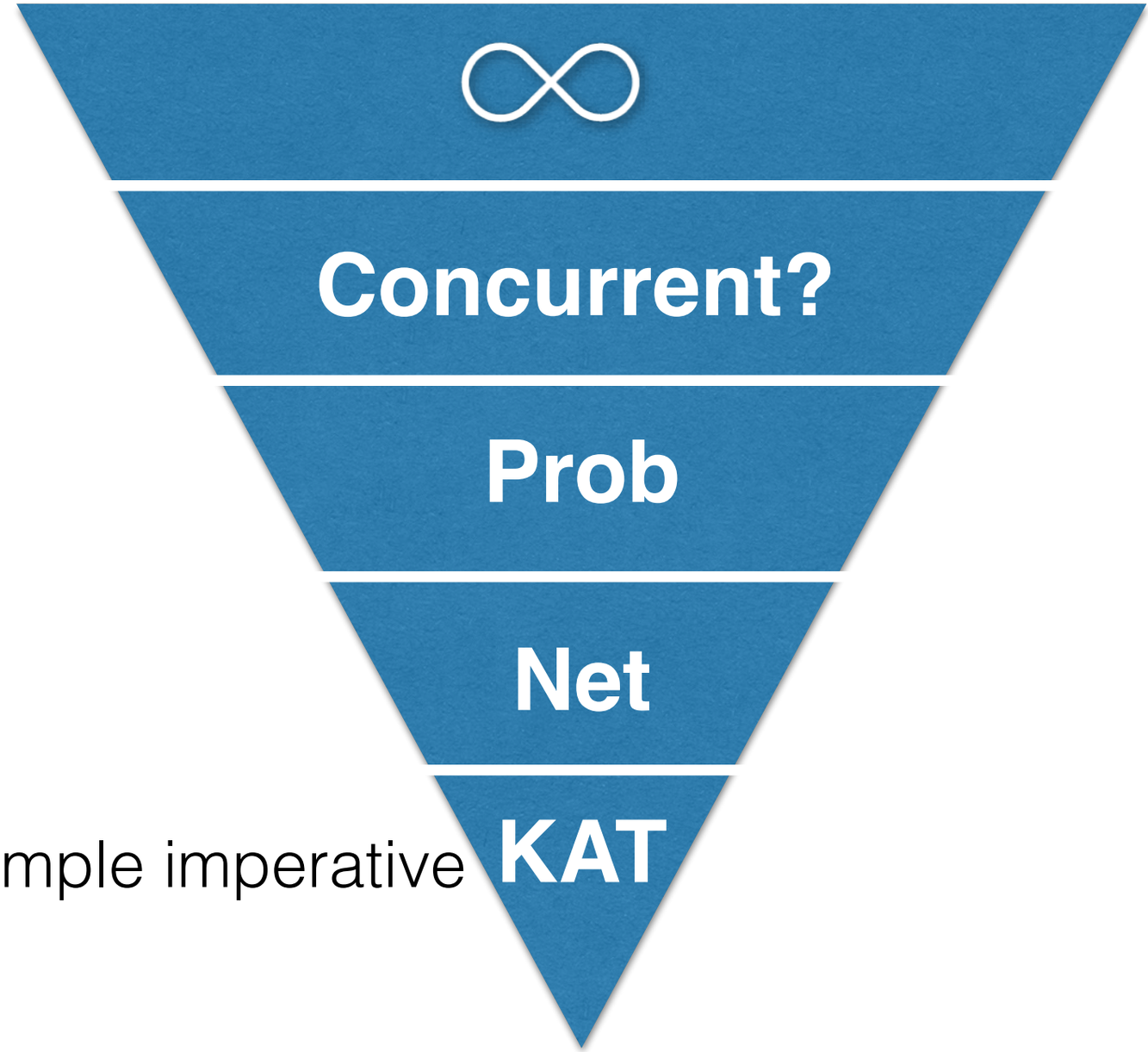
Scalable verification

Fundamental correspondence to automata and language theory

The KA(T) tower principle

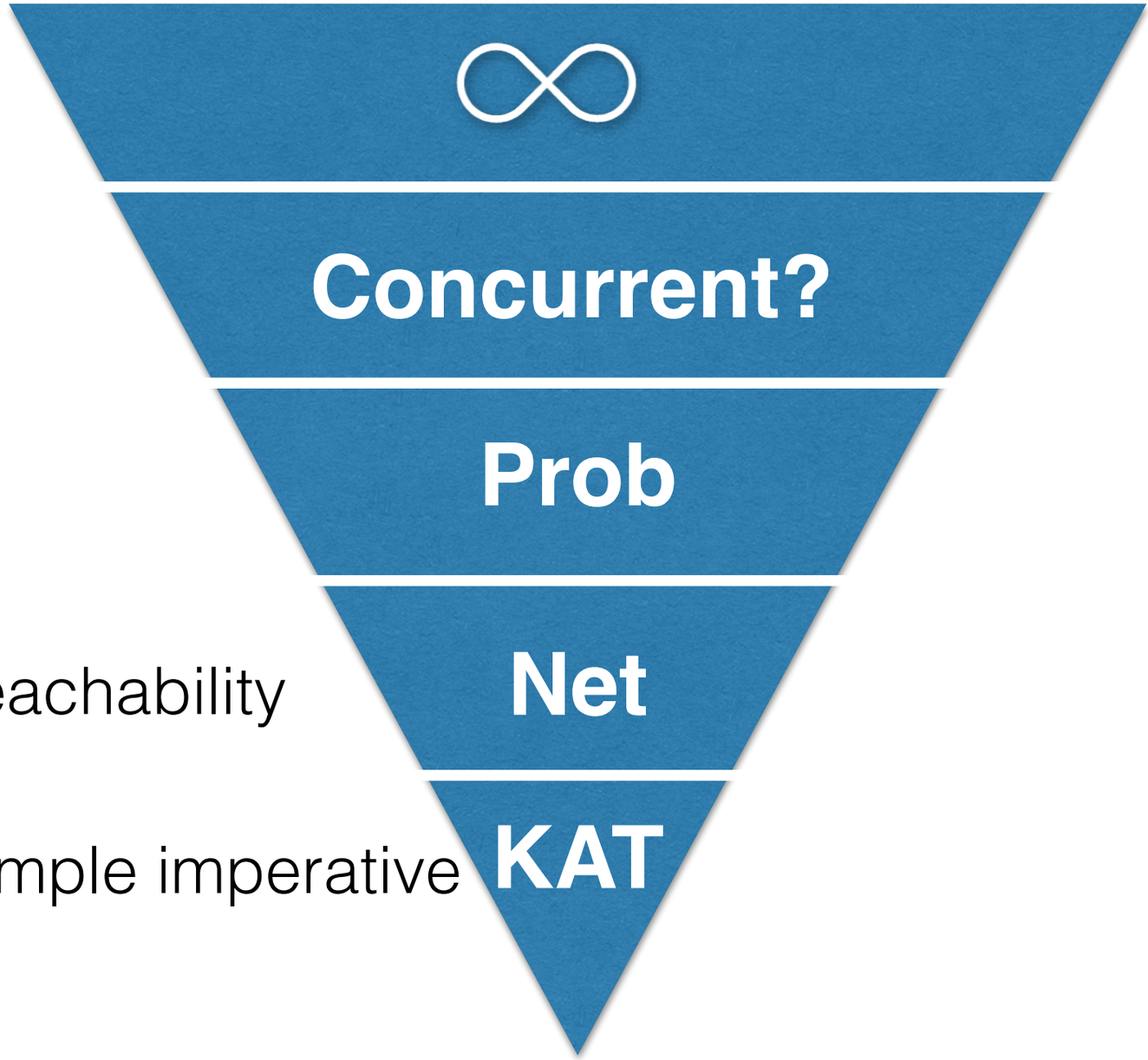


The KA(T) tower principle

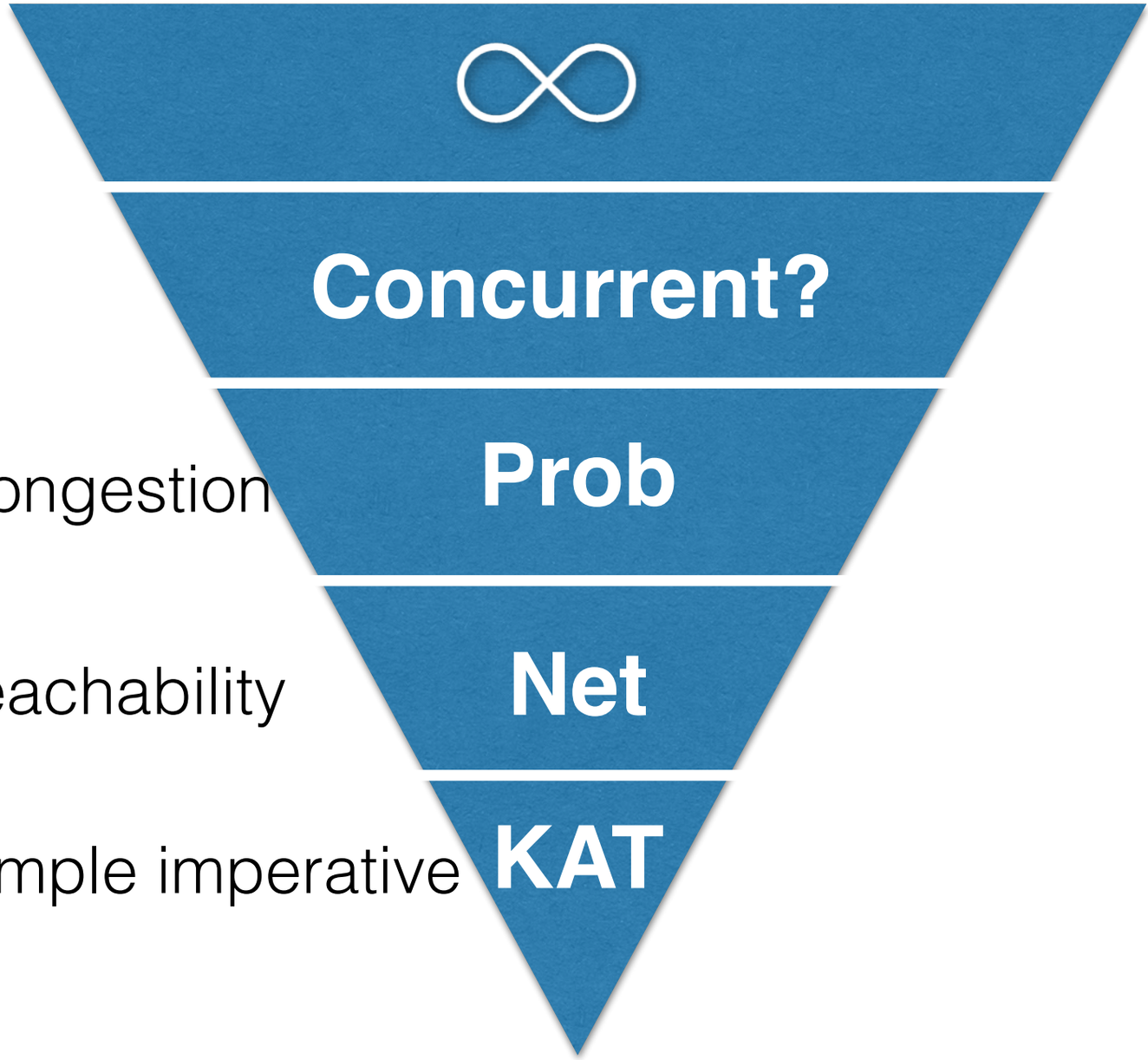


simple imperative

The KA(T) tower principle



The KA(T) tower principle



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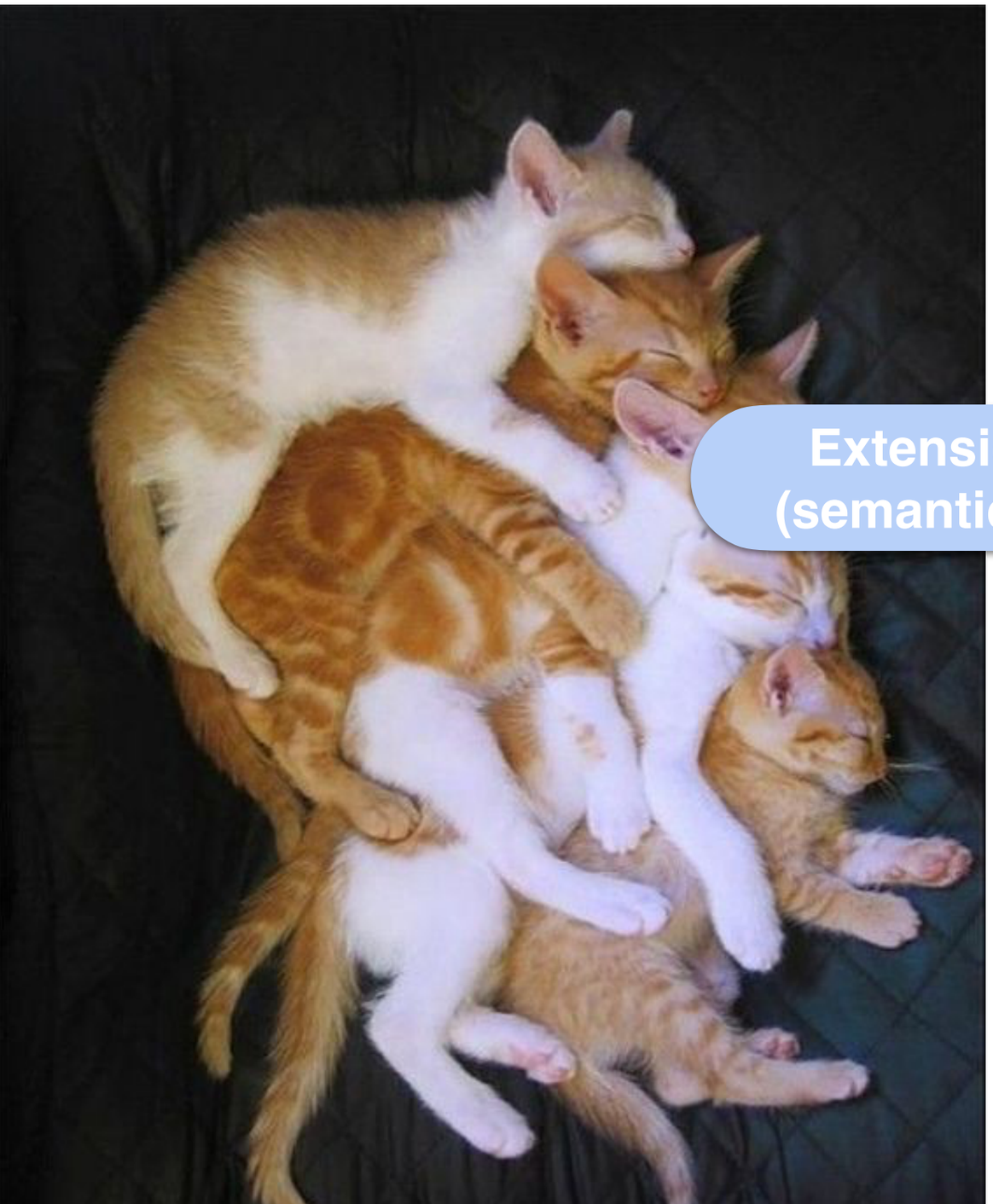


Extensions conservative
(semantics and algorithms)

∞

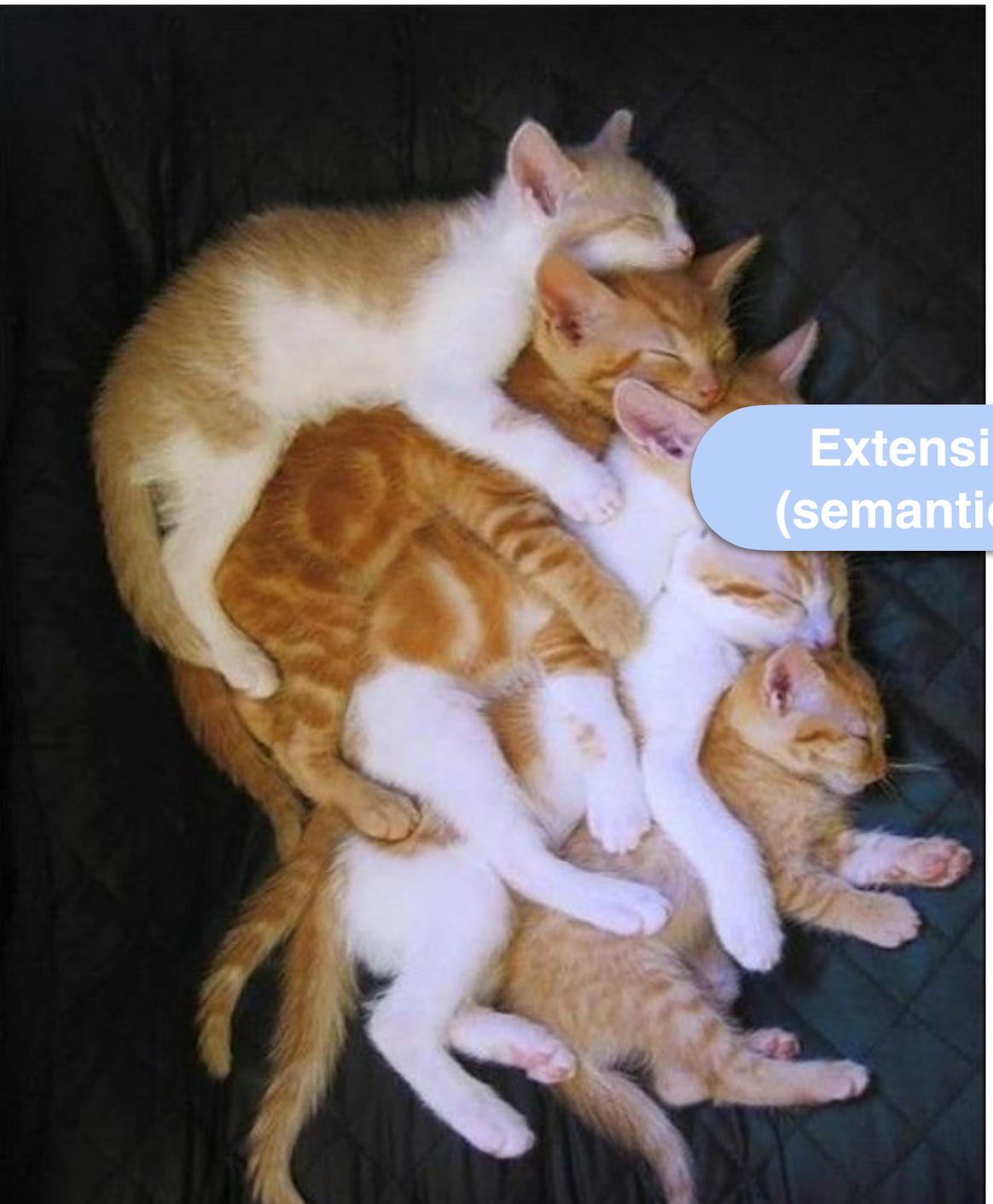
Concurrent?

Prob

Net

KAT

The KA(T) tower principle



Extensions conservative
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∞

Concurrent?

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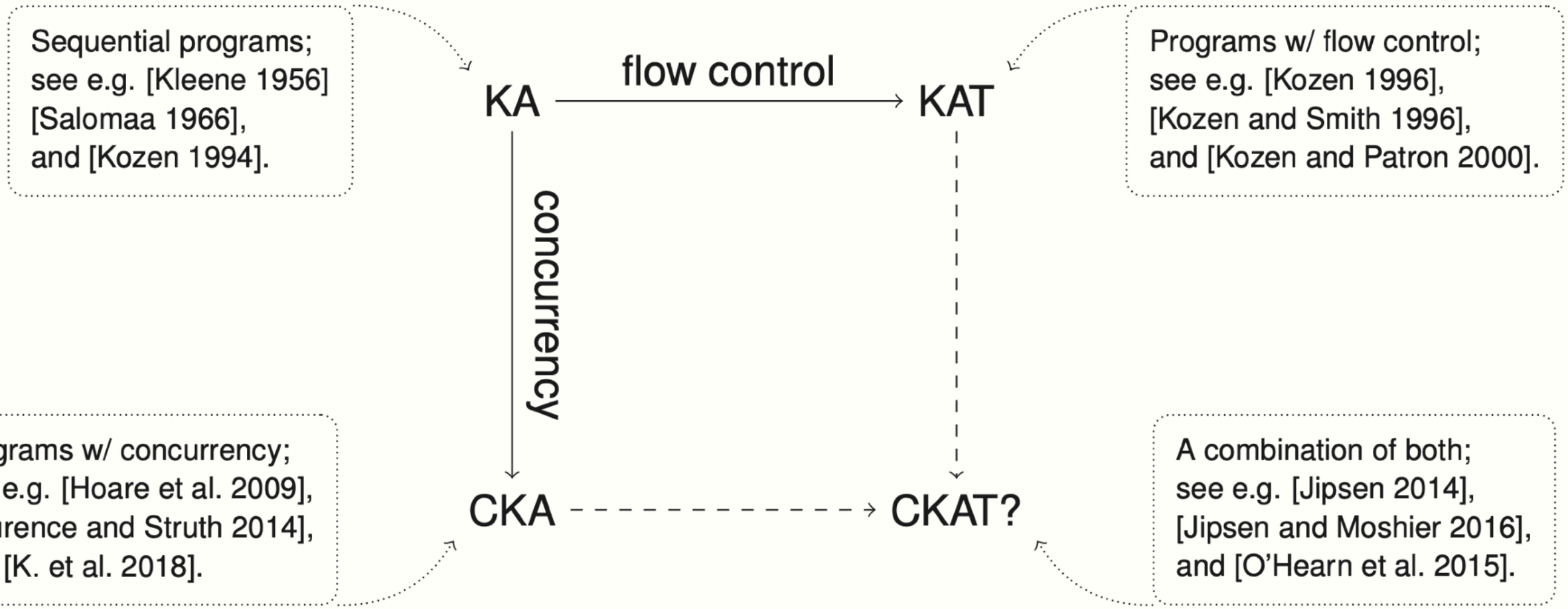
Net

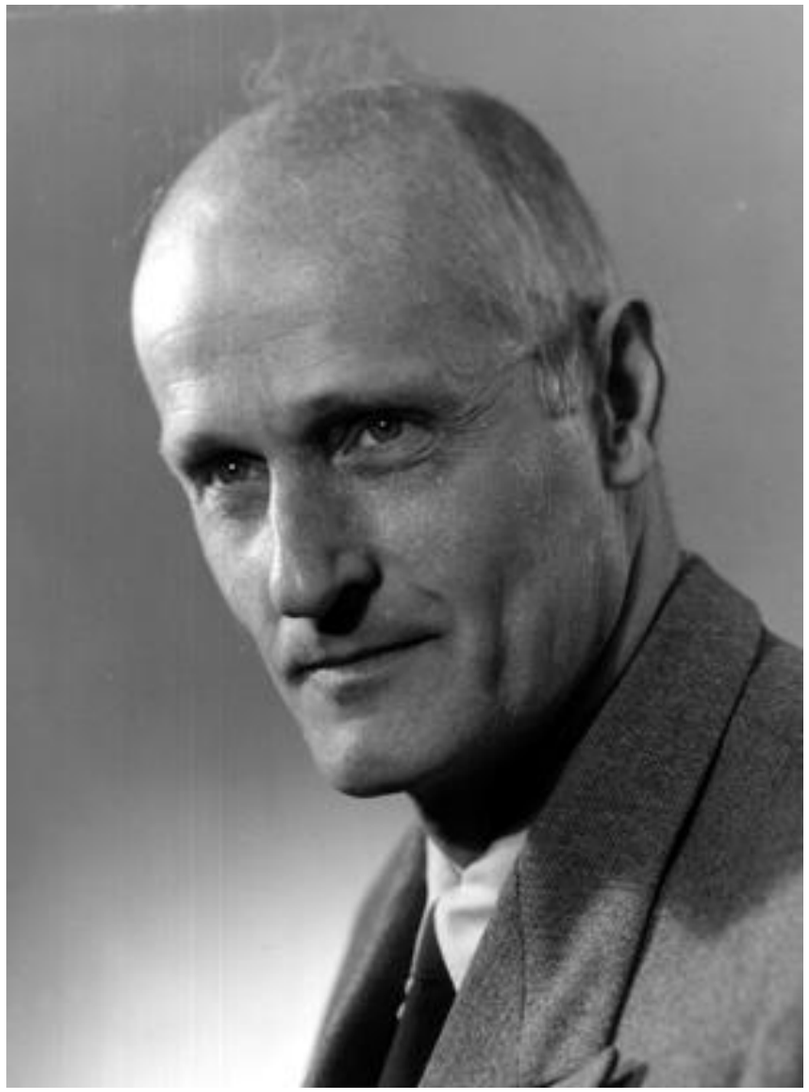
CKAT

This talk


Sequential programs;
see e.g. [Kleene 1956]
[Salomaa 1966],
and [Kozen 1994].
KA
flow control
KAT
Programs w/ flow control;
see e.g. [Kozen 1996],
[Kozen and Smith 1996],
and [Kozen and Patron 2000].
concurrency
Programs w/ concurrency;
see e.g. [Hoare et al. 2009],
[Laurence and Struth 2014],
and [K. et al. 2018].
CKA
CKAT?
A combination of both;
see e.g. [Jipsen 2014],
[Jipsen and Moshier 2016],
and [O'Hearn et al. 2015].

Let’s start from the beginning…

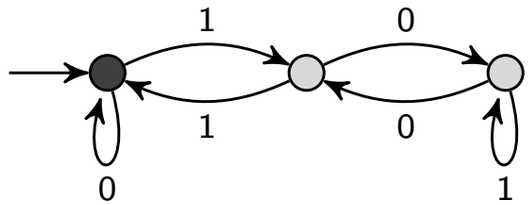
Kleene algebra - **KA**



Stephen Cole Kleene
(1909–1994)

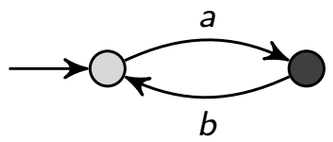
$(0 + 1(01^*0)^*1)^*$

{multiples of 3 in binary}



$(ab)^*a = a(ba)^*$

$\{a, aba, ababa, \ldots\}$



$(a + b)^* = a^*(ba^*)^*$

{all strings over $\{a, b\}$}

Kleene algebra - **KA**

program	expression
atomic action	$a, b, \ldots \in \Sigma$
abort execution	0
no-operation	1
nondeterministic choice	$e + f$
sequential composition	$e \cdot f$
repetition	e^*

Kleene algebra - **KA**

$$e + 0 \equiv e \qquad e + e \equiv e \qquad e + f \equiv f + e \qquad e + (f + g) \equiv (e + f) + g$$

Kleene algebra - **KA**

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Kleene algebra - **KA**

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$$1 + e \cdot e^* \equiv e^* \qquad e \cdot f + g \leqq f \implies e^* \cdot g \leqq f$$

Kleene algebra - **KA**

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Theorem [Kozen'90]

KA axioms are sound and complete

Equivalence

Via Completeness
using KA axioms

Via Kleene's Theorem
using automata

Near-linear algorithms
Hopcroft-Karp, Bisimulation up-to

Equivalence

Via Completeness
using KA axioms

Via Kleene's Theorem
using automata

Near-linear algorithms
Hopcroft-Karp, Bisimulation up-to

Equivalence is essential to many verification questions!

RESEARCH-ARTICLE FREE ACCESS

NetKAT: semantic foundations for networks

Authors: Carolyn Jane Anderson, Nate Foster, Arjun Guha, Jean-Baptiste Jeannin, Dexter Kozen, Cole Schlesinger, David Walker Authors Info & Affiliations

Equivalence

Via Completeness
using KA axioms

Via Kleene's Theorem
using automata

Near-linear algorithms
Hopcroft-Karp, Bisimulation up-to

Equivalence is essential to many verification questions!

Efficient fragments are still being studied!

Equivalence

Via Completeness
using KA axioms

Via Kleene's Theorem
using automata

Near-linear algorithms
Hopcroft-Karp, Bisimulation up-to

Equivalence is essential to many verification questions!

Efficient fragments are still being studied!

ARTICLE OPEN ACCESS

Guarded Kleene algebra with tests: verification of uninterpreted programs in nearly linear time

Authors: Steffen Smolka, Nate Foster, Justin Hsu, Tobias Kappé, Dexter Kozen, Alexandra Silva Authors Info & Affiliations

KAT

$(K, B, +, \cdot, ^*, \bar{}, 0, 1), \quad B \subseteq K$

- $(K, +, \cdot, ^*, 0, 1)$ is a Kleene algebra
- $(B, +, \cdot, \bar{}, 0, 1)$ is a Boolean algebra
- $(B, +, \cdot, 0, 1)$ is a subalgebra of $(K, +, \cdot, 0, 1)$

- $p, q, r, \ldots$ range over K
- $a, b, c, \ldots$ range over B

KAT

KAT = simple imperative language

If b **then** p **else** q = b;p + !b;q

While b **do** p = (bp)*!b

Boolean algebra

KAT

KAT = simple imperative language

If b **then** p **else** q = b;p + !b;q

While b **do** p = (bp)*!b

KAT

KAT = simple imperative language

If b **then** p **else** q = b;p + !b;q

While b **do** p = (bp)*!b

$$[\![p]\!] \subseteq At \cdot (\Sigma \cdot At)^*$$

KAT

$$[\![p]\!] \subseteq At \cdot (\Sigma \cdot At)^*$$

If b **then** p **else** q = b;p + !b;q

While b **do** p = (bp)*!b

$$\{\alpha_b \mathsf{p} \beta, \alpha_{!b} \mathsf{p} \beta\}$$

$$\{\alpha_{!b}, \alpha_b \mathsf{p} \alpha_{!b}, \alpha_b \mathsf{p} \alpha_b \mathsf{p} \alpha_{!b}, \ldots\}$$

valuations
variables

Equivalence

Theorem [Kozen'96]

KA+BA axioms are sound and complete

Via Completeness
using KAT axioms

Via Kleene's Theorem
using *guarded* automata

Equivalence

Theorem [Kozen'96]

KA+BA axioms are sound and complete

Via Completeness
using KAT axioms

Via Kleene's Theorem
using *guarded* automata

$$\{b\}\, p\, \{c\} \overset{\triangle}{\iff} bp \leqslant pc$$
$$\iff bp = bpc$$
$$\iff bp\bar{c} = 0$$

Concurrent programs

initial state: x = 0, y = 0	
T0	T1
a: x ← 1;	c: y ← 1;
b: r1 ← y;	d: r2 ← x;
assert: $r_1 \neq 0$ xor $r_2 \neq 0$	

Concurrent programs

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Can we reason about these programs (co)algebraically?

Concurrent programs

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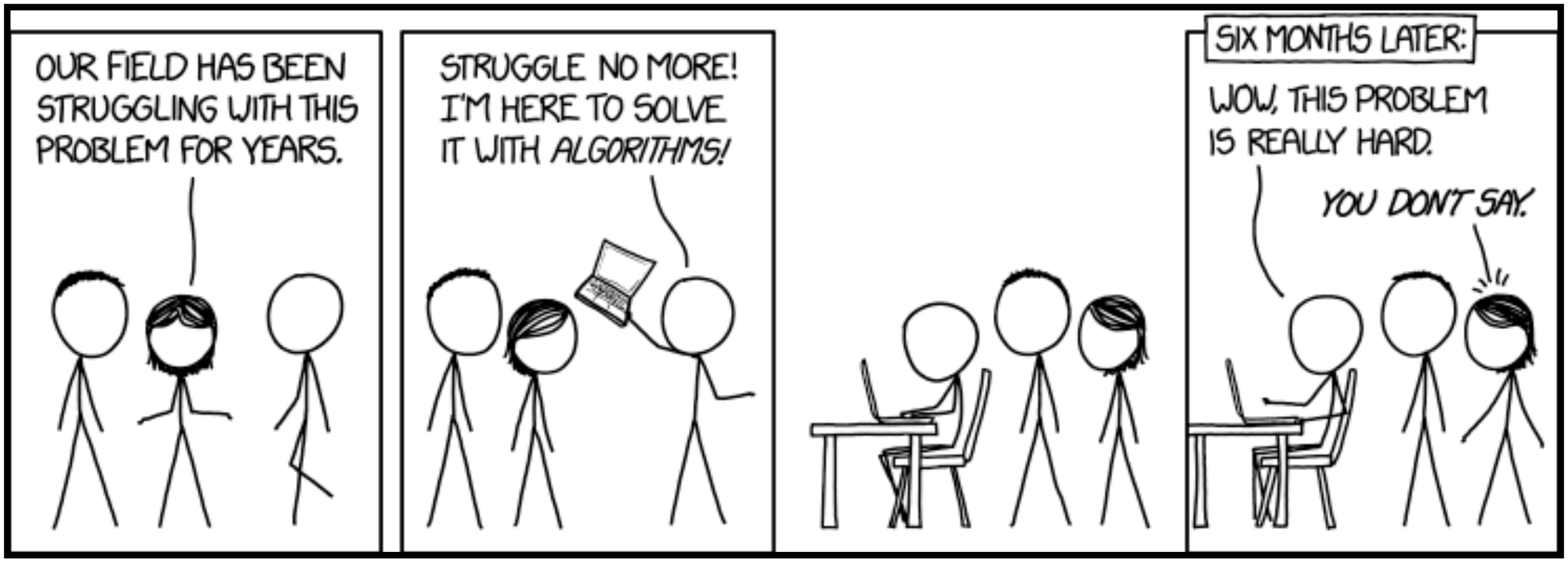
Can we reason about these programs (co)algebraically?

$$(x = 0 \wedge y = 0) \cdot (T0 \parallel T1) \leq \neg(r1 = 0 \wedge r2 = 0)$$

Concurrency



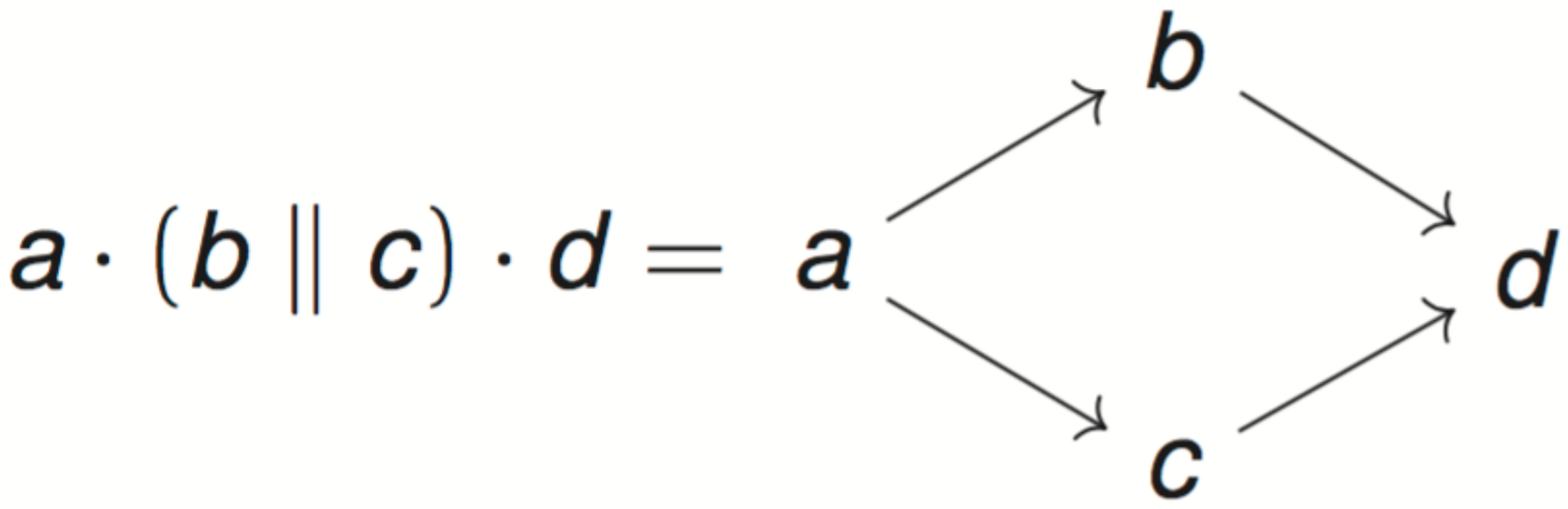
Concurrent Kleene Algebra
(Hoare, Moeller, Struth, Wehrman'09)

$$p \;||\; q$$

Concurrent Kleene Algebra
(Hoare, Moeller, Struth, Wehrman'09)

$$p \mid\mid q$$

Semantics = languages of PomSets



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Kleene Theorem

Axiomatisation

Decision Procedure

Concurrent Kleene Algebra
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Concurrent Kleene Algebra
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$$e \parallel f \equiv f \parallel e \qquad e \parallel (f \parallel g) \equiv (e \parallel f) \parallel g$$

$$e \parallel 1 \equiv e \qquad e \parallel 0 \equiv 0 \qquad e \parallel (f + g) \equiv e \parallel f + e \parallel g$$

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$$e \parallel 1 \equiv e \qquad e \parallel 0 \equiv 0 \qquad e \parallel (f + g) \equiv e \parallel f + e \parallel g$$

$$(e \parallel g) \cdot (f \parallel h) \leqq (e \cdot f) \parallel (g \cdot h). \qquad \textbf{Exchange law}$$

Concurrent Kleene Algebra
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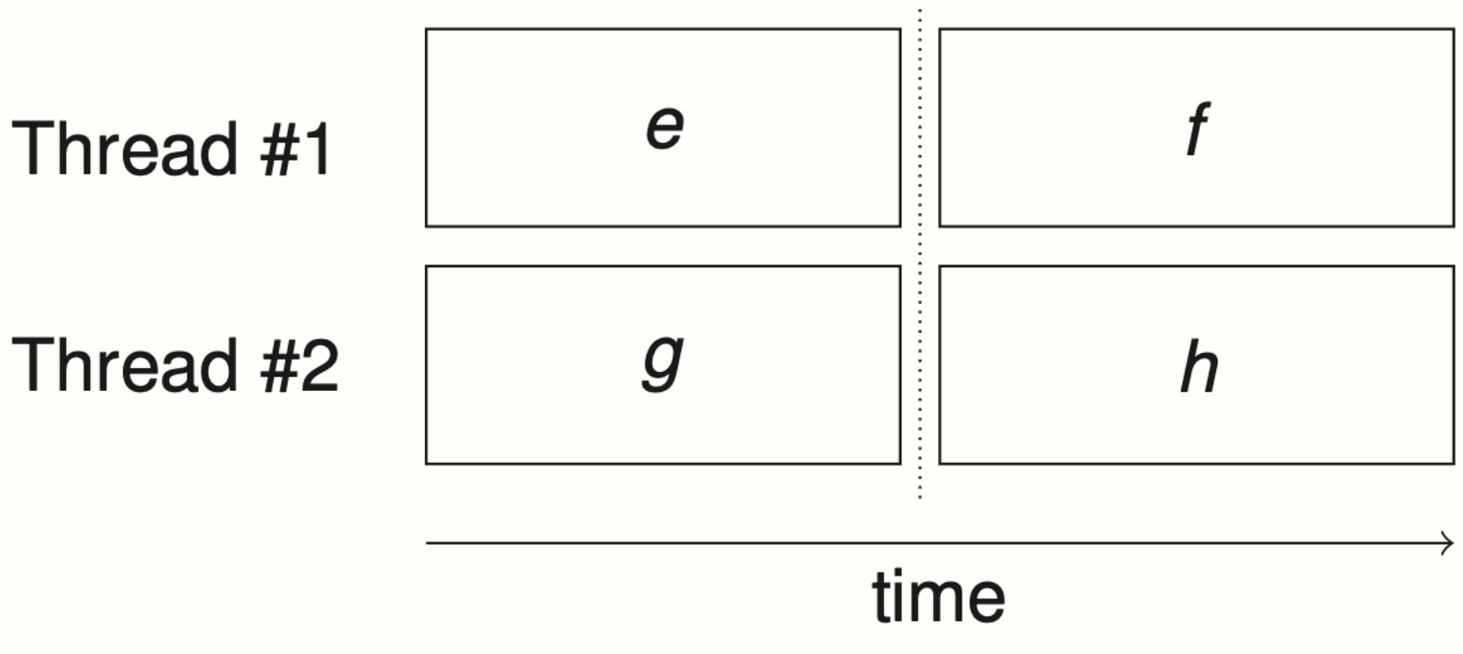

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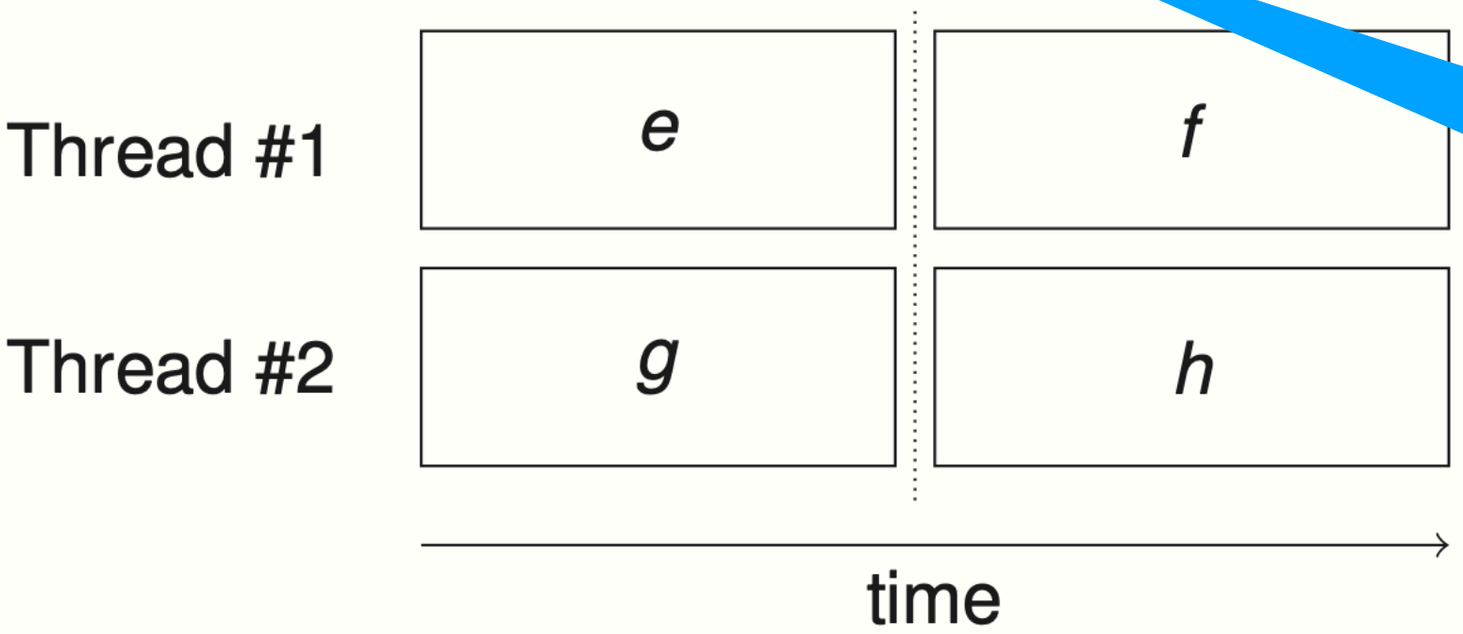

Interleaving as special case: p||q=p.q+q.p


GoNeCo

Kleene Theorem

Axiomatisation

Decision Procedure

GoNeCo

Brzozowski goes concurrent: a Kleene theorem for Pomset languages (CONCUR 2017)

Tobias Kappé, Paul Brunet, Bas Luttik, Alexandra Silva, and Fabio Zanasi

Kleene Theorem

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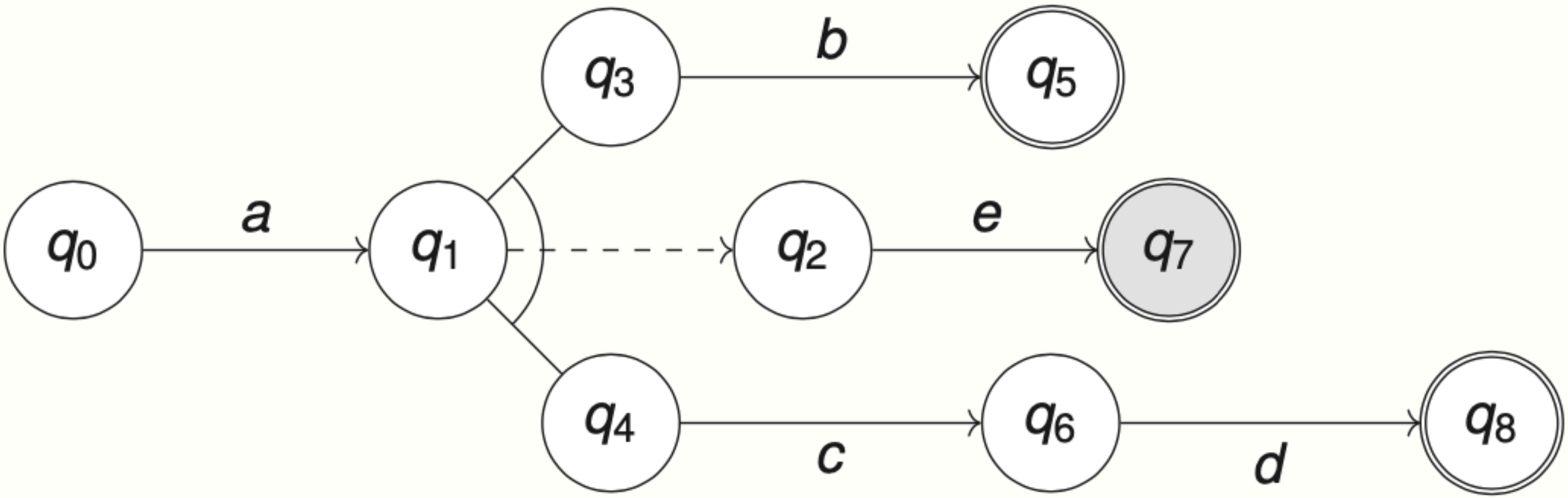
Concurrent Kleene Algebra: free model and completeness (ESOP 2018)

Tobias Kappé, Paul Brunet, Alexandra Silva, and Fabio Zanasi

Kleene Theorem

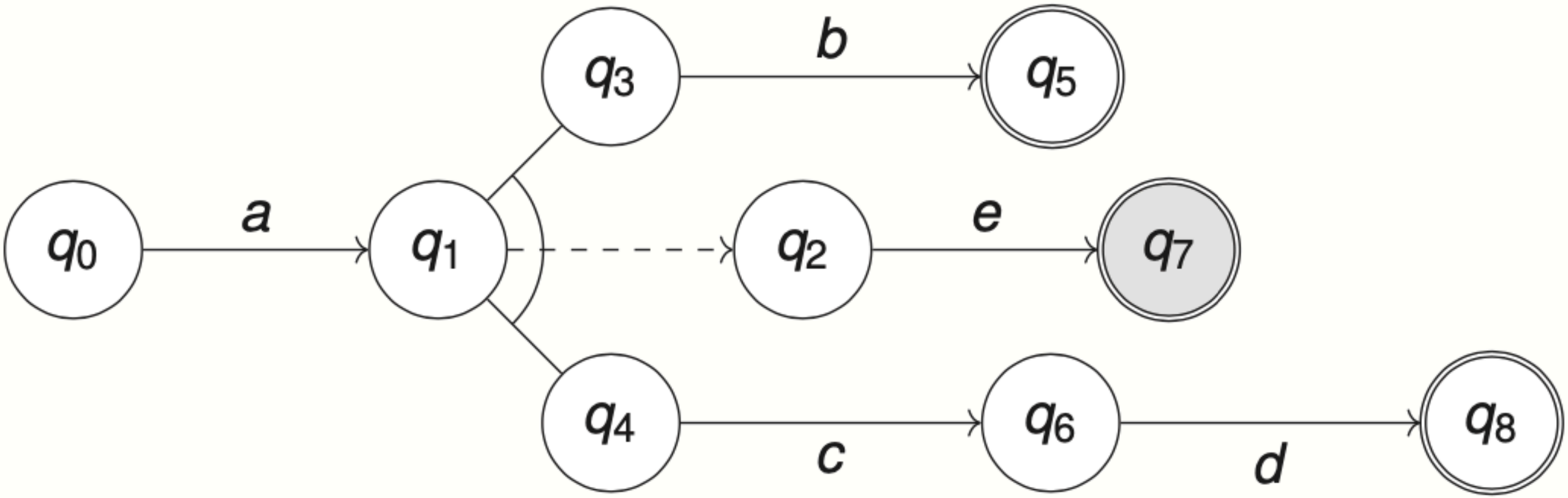
Fork Automata (Gischer'88)



$$a(b \parallel cd)e$$

Kleene Theorem

Fork Automata (Gischer'88)



Kleene Theorem

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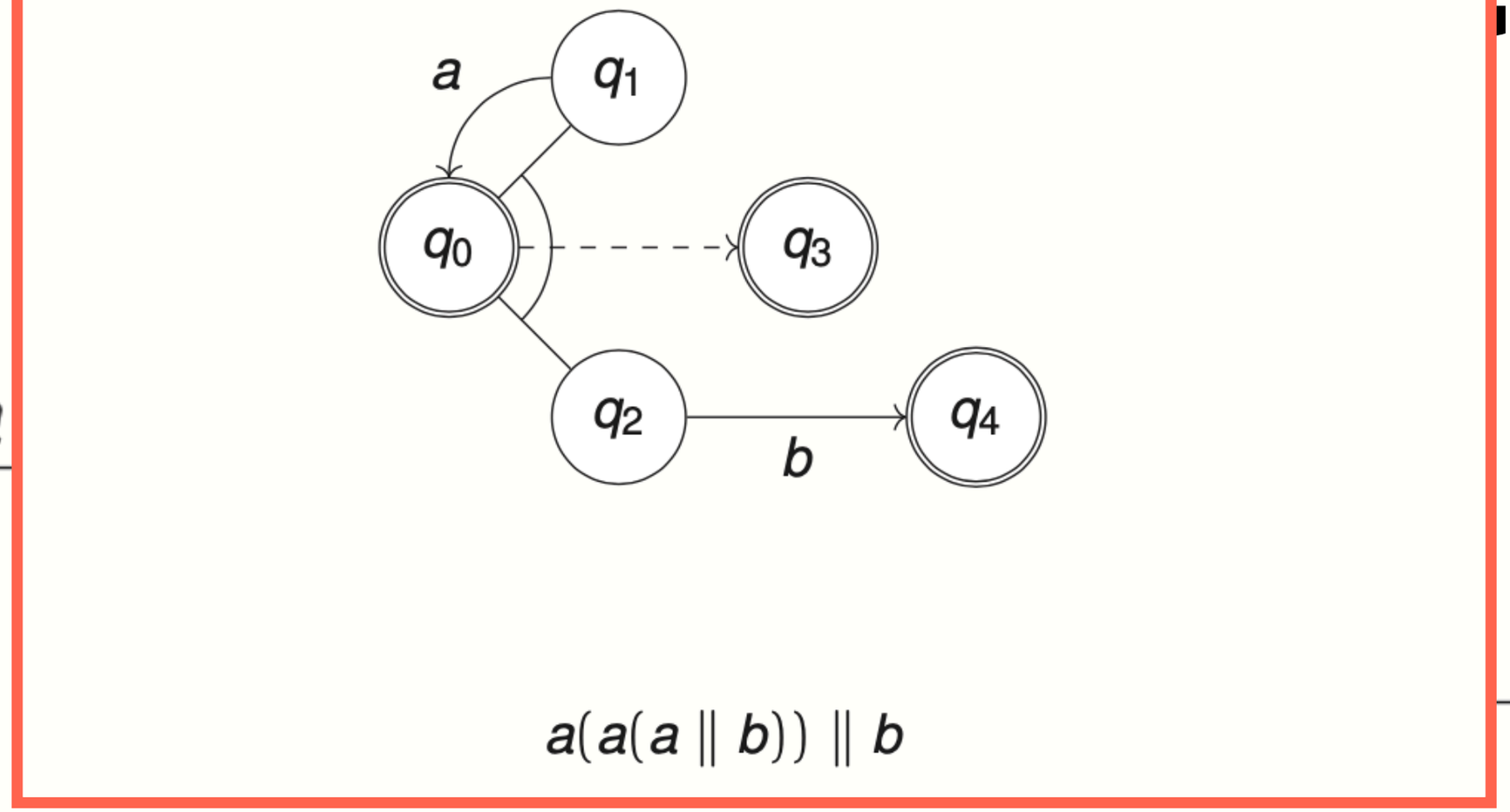

Brzozowski-like construction

Automata

Expressions

Requires restriction

CKA ✓

CKAT

CKA

next

GoNeCo

CNetKAT — soon

CKAT — next

CKA 

CKAT [Jipsen'14]

$$e, f ::= \underbrace{p \mid 0 \mid 1 \mid a \in \Sigma \mid e + f \mid e \cdot f \mid e^*}_{\text{KAT terms}} \mid e \parallel f$$

$$p, q ::= \bot \mid \top \mid t \in T \mid p \vee q \mid p \wedge q \mid \overline{p}$$

CKAT [Jipsen'14]

CKA terms

$$e, f ::= p \mid 0 \mid 1 \mid a \in \Sigma \mid e + f \mid e \cdot f \mid e^* \mid e \parallel f$$

KAT terms

$$p, q ::= \bot \mid \top \mid t \in T \mid p \vee q \mid p \wedge q \mid \overline{p}$$

The problem with CKAT

$$
\begin{aligned}
p \cdot e \cdot \overline{p} &\leqq (p \parallel e) \cdot \overline{p} \\
&\equiv (p \parallel e) \cdot (\overline{p} \parallel 1) \\
&\leqq (p \cdot \overline{p}) \parallel (e \cdot 1) \\
&\equiv (p \cdot \overline{p}) \parallel e
\end{aligned}
$$

The problem with CKAT

$$
\begin{aligned}
p \cdot e \cdot \overline{p} &\leqq (p \parallel e) \cdot \overline{p} \\
&\equiv (p \parallel e) \cdot (\overline{p} \parallel 1) \\
&\leqq (p \cdot \overline{p}) \parallel (e \cdot 1) \\
&\equiv (p \cdot \overline{p}) \parallel e \\
&\equiv (p \wedge \overline{p}) \parallel e \\
&\equiv \bot \parallel e \\
&\equiv 0 \parallel e \\
&\equiv 0
\end{aligned}
$$

The problem with CKAT

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\end{aligned}
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&\equiv 0 \parallel e \\
&\equiv 0
\end{aligned}
$$

Every test is an invariant of every program!

Crossroads

Non-congruence?

Drop $e \parallel 0 \equiv 0$?

Drop exchange law?

Crossroads

See also [Bergstra and Ponse 2011].

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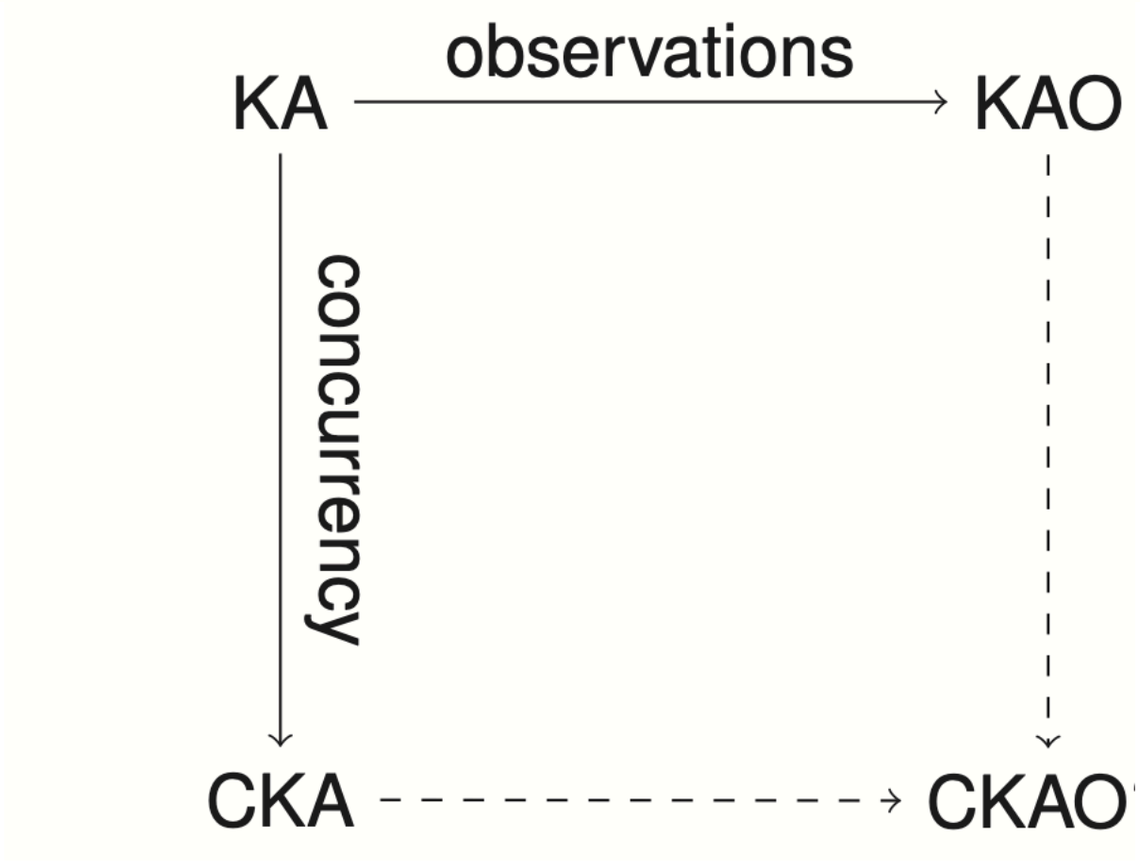
A new algebra

$$p \wedge q \equiv p \cdot q \rightsquigarrow p \wedge q \leqq p \cdot q$$

~~$1 \equiv \top$~~



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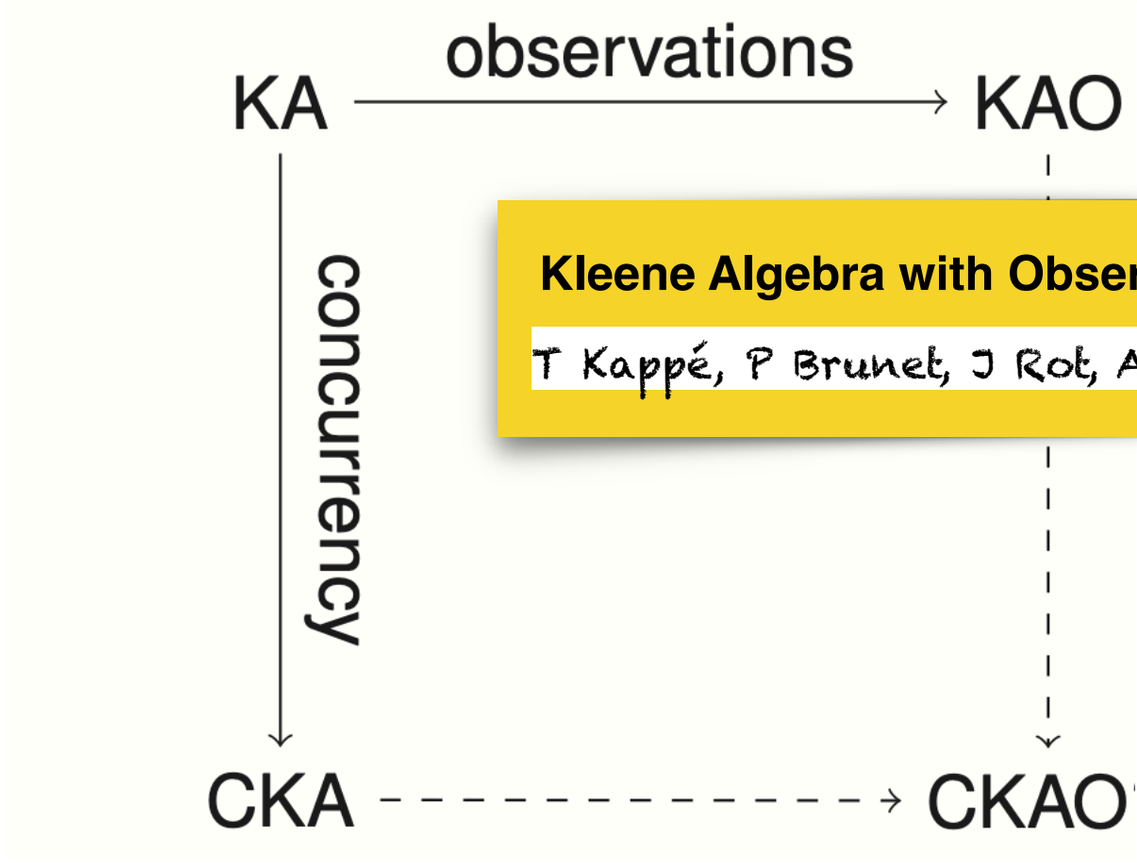

Kleene Algebra with Observations (CONCUR'19)

T Kappé, P Brunet, J Rot, A Silva, J Wagemaker, and F Zanasi

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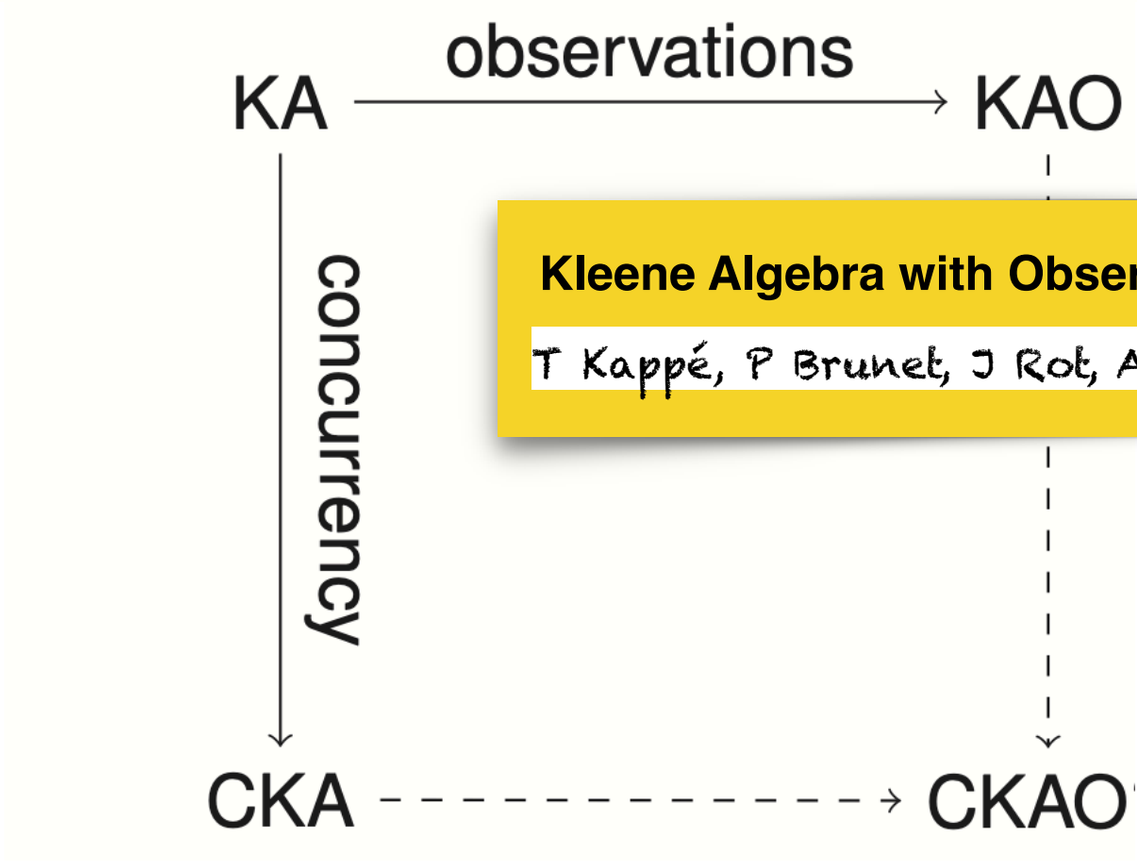

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Concurrent Kleene Algebra with Observations: from Hypotheses to Completeness (Fossacs'20)

T Kappé, P Brunet, A Silva, J Wagemaker, and F Zanasi

CKAO

initial state: x = 0, y = 0	
T0	T1
a: x ← 1;	c: y ← 1;
b: r1 ← y;	d: r2 ← x;
assert: 0:r1 = 0 ∧ 1:r2 = 0	

Can we reason about these programs (co)algebraically?

$$(x = 0 \wedge y = 0) \cdot (T0 \mid\mid T1) \leq \neg(r1 = 0 \wedge r2 = 0)$$

Axioms
KA+BA+CKA

Keep conjunction and sequential composition distinct

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Axioms
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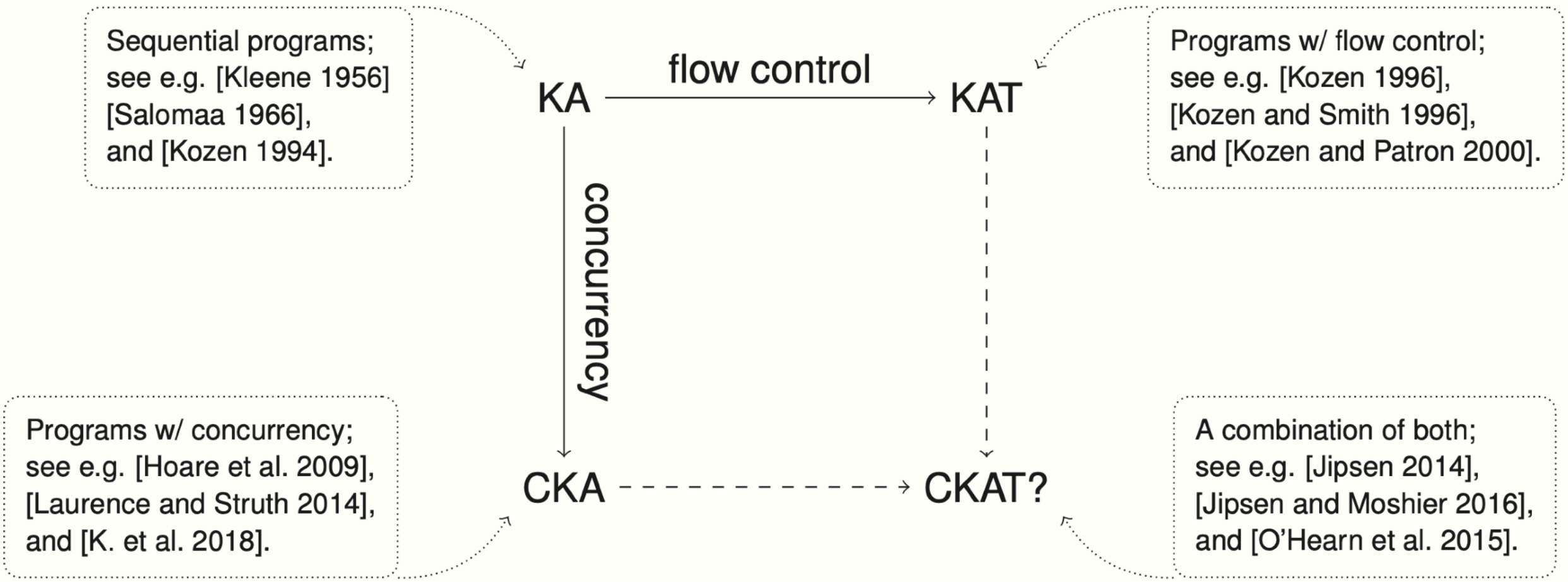
Keep conjunction and sequential composition distinct

Theorem 5.6 (Soundness and Completeness CKAO). *Let* $e, f \in \mathcal{T}_{\mathsf{CKAO}}$. *The following holds:*

$$e \equiv_{\mathsf{CKAO}} f \Leftrightarrow [\![e]\!]_{\mathsf{CKAO}} = [\![f]\!]_{\mathsf{CKAO}}.$$

Moreover, given $e, f \in \mathcal{T}$, *it is decidable whether* $[\![e]\!]_{\mathsf{CKAO}} = [\![f]\!]_{\mathsf{CKAO}}$.

Conclusions



Conclusions

Sequential programs;
see e.g. [Kleene 1956]
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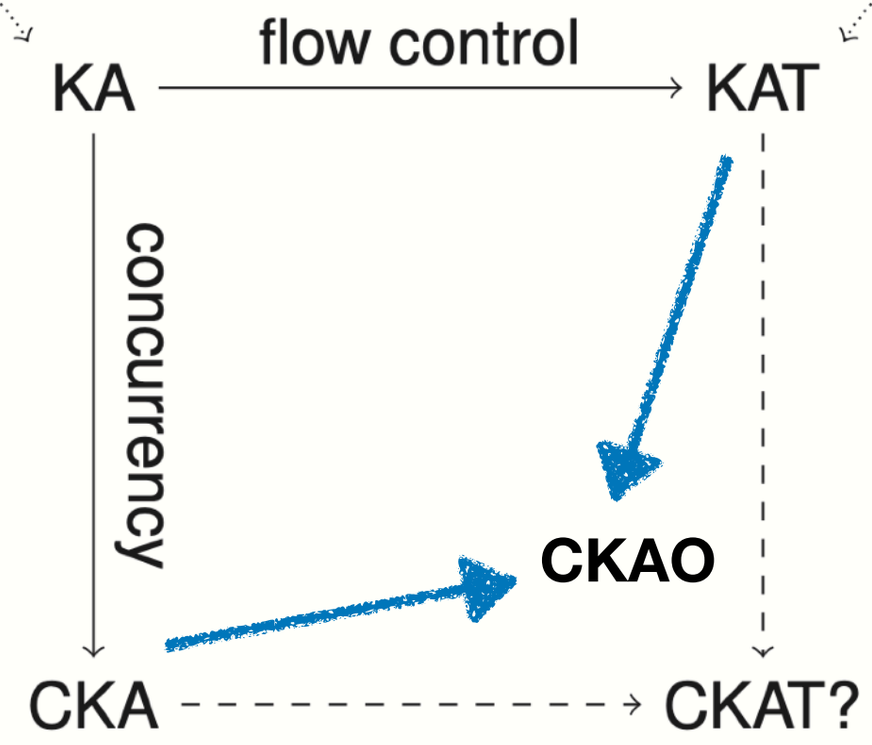

A (co)algebraic framework to analyse programs with concurrency and control flow

Programs w/ concurrency;
see e.g. [Hoare et al. 2009],
[Laurence and Struth 2014],
and [K. et al. 2018].

A combination of both;
see e.g. [Jipsen 2014],
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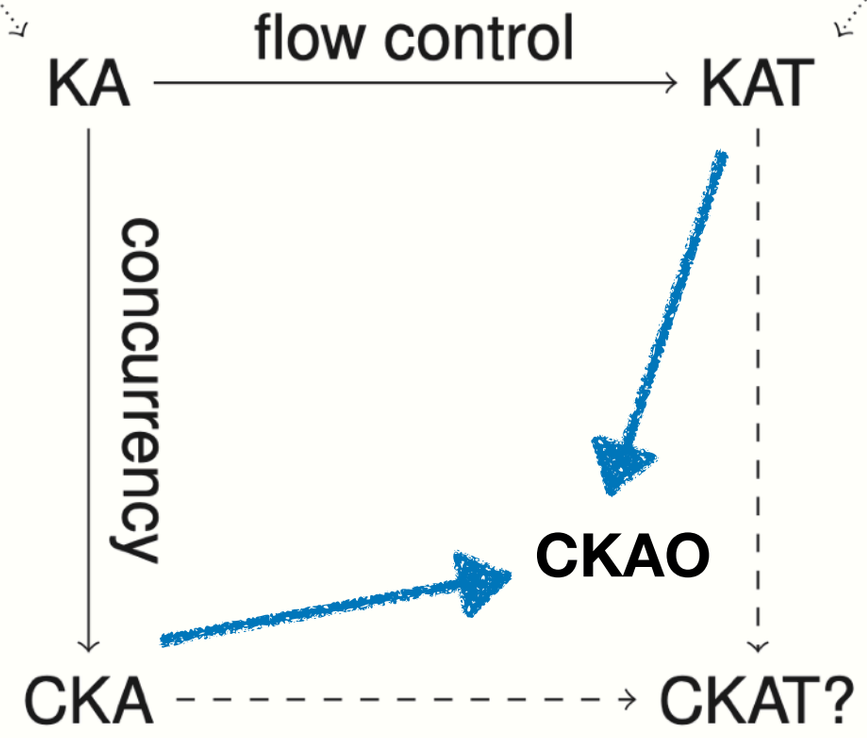

A (co)algebraic framework to analyse programs with concurrency and control flow

Sound and complete axiomatization
Decidability

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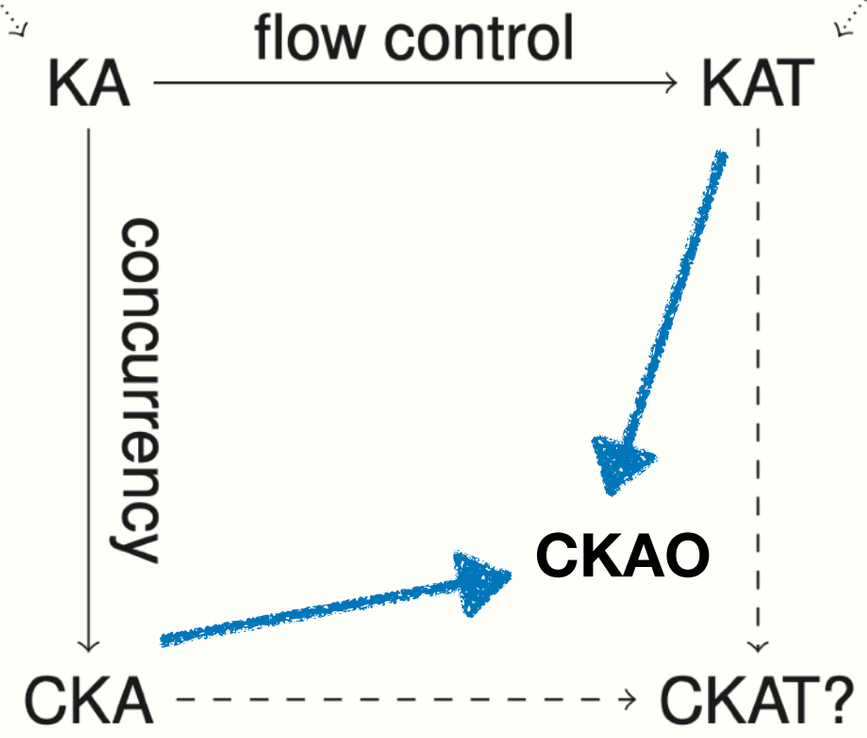

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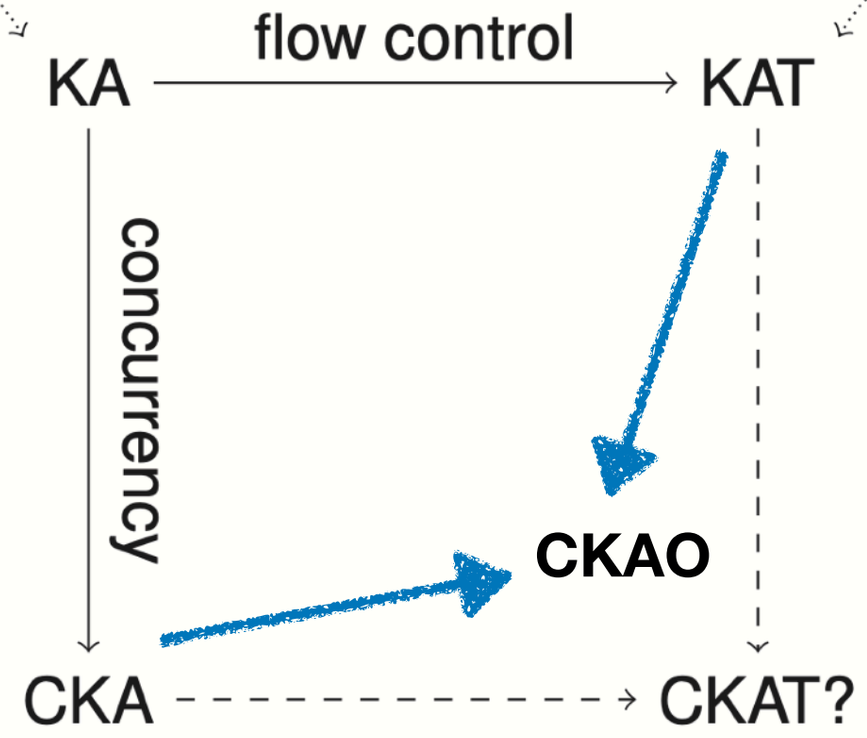

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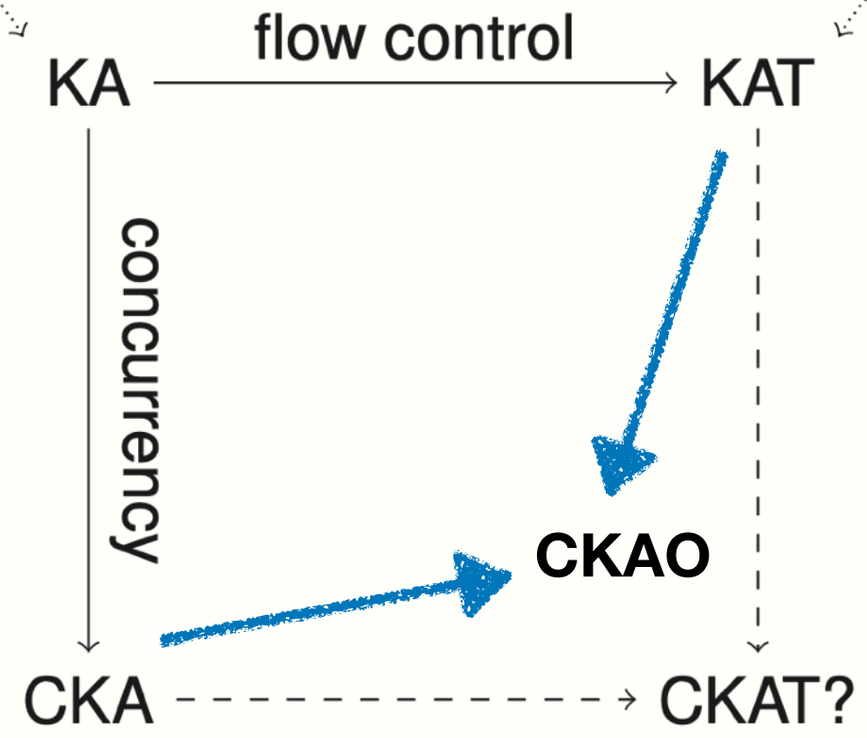

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Questions?

Credits

GoNeCo

http://coneco-project.org