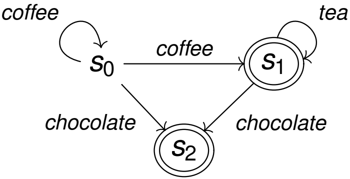
Determinization constructions: from automata to coalgebras

Alexandra Silva
joint work with F. Bonchi, M. Bonsangue and J. Rutten

ACG, November 2011

Motivation (by example)

Non-deterministic automata



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chocolate	✓
coffee;chocolate	✓
coffee;tea	✓
coffee;coffee;tea	✓
chocolate;tea	×

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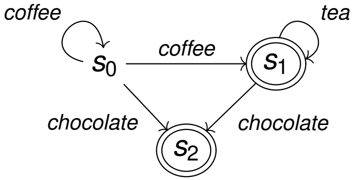

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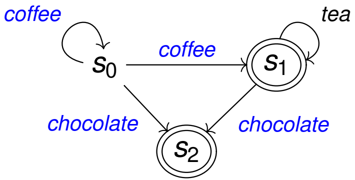

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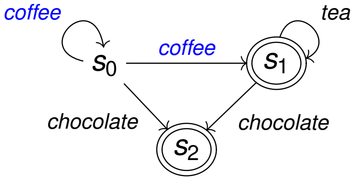

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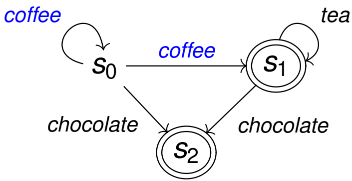

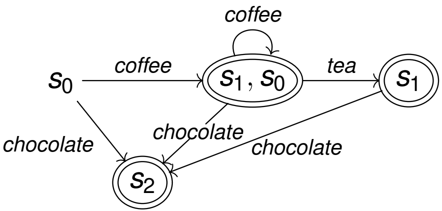

The subset construction

Starting point: Non-deterministic automaton
$\mathcal{A} = (Q, \langle o, \delta \rangle \colon Q \to 2 \times \mathcal{P}(Q)^A)$

Goal: Deterministic automaton
$det(\mathcal{A}) = (\mathcal{P}(Q), \langle \hat{o}, \hat{\delta} \rangle \colon \mathcal{P}(Q) \to 2 \times \mathcal{P}(Q)^A)$

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$$\hat{o}(Y) = \begin{cases} 1 & \exists_{y \in Y} o(y) = 1 \\ 0 & \text{otherwise} \end{cases} \qquad \hat{\delta}(Y)(a) = \bigcup_{y \in Y} \delta(y)(a).$$

This construction guarantees that any word which labels a successful path in $\mathcal{A}$ also labels a successful path in $det(\mathcal{A})$ (and vice-versa).

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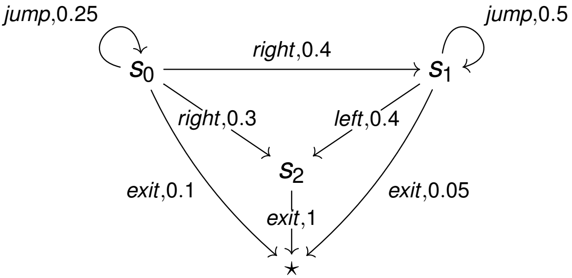

What is the probability that you jump, go right and then exit?

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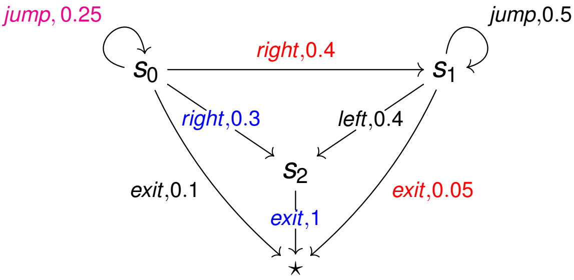

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Motivation (by another example)

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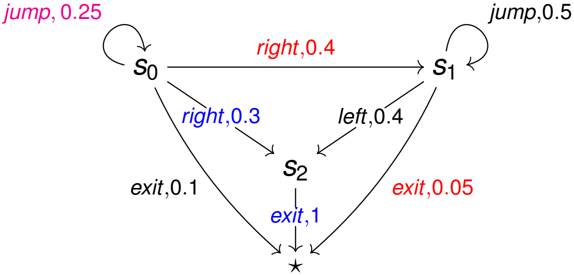

What is the probability that you jump, go right and then exit?

$$0.25\times0.4\times0.05+0.25\times0.3\times1 = 0.08$$

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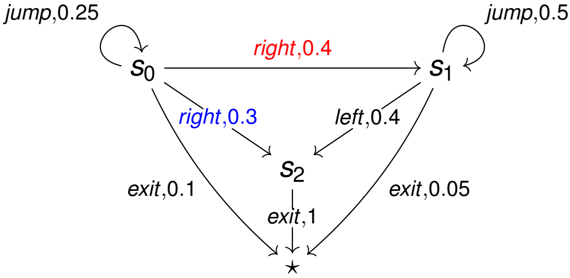
Weighted automata: how to eliminate *choice*?



$$s_0 \xrightarrow{right} 0.4s_1 + 0.3s_2$$

$$s_0 \xrightarrow{jump} 0.25s_0$$

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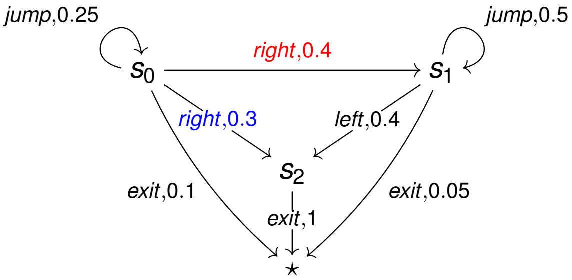

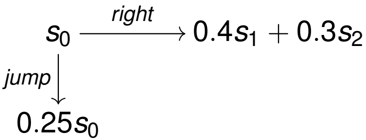

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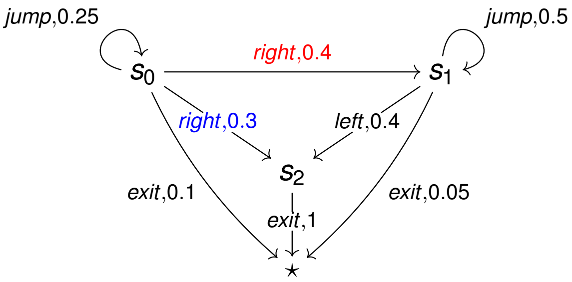

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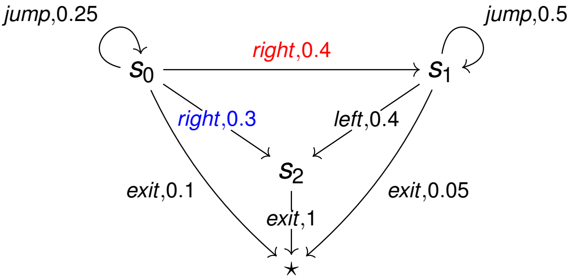

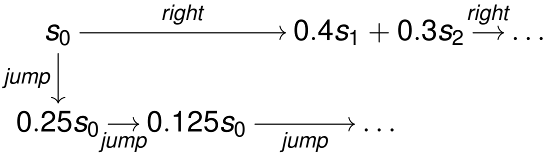

The linearization construction

Starting point: weighted automaton $\mathcal{A} = (Q, \langle o, \delta \rangle \colon Q \to \mathbb{R} \times V(Q)^A)$, $V(S) = \mathbb{R} \to S =$ linear combinations of S.

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- In non-deterministic automata we go from a finite automaton to a finite deterministic automaton where states are sets of the original states.
- In weighted automata we go from a finite automaton to an infinite deterministic automaton where states are linear combinations of the original states.
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What do these constructions have in common?

- First: what do NDA and WA have in common?

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Coalgebras: $(S, S \to T(S))$.

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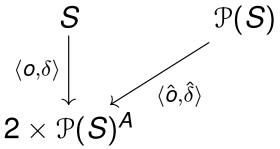

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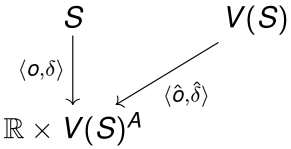

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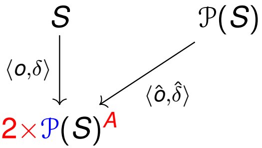

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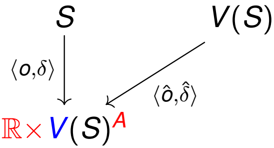
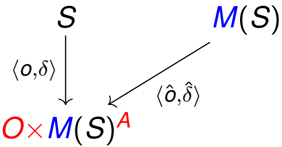
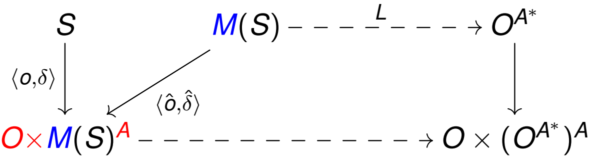




Recall from Jan's talk: semantics by finality



How to tie things together: construction and semantics?

$$L(Q) = \bigcup_{q \in Q} L(q) \qquad L(rq_1 + sq_2) = rL(q_1) + sL(q_2)$$

Language semantics

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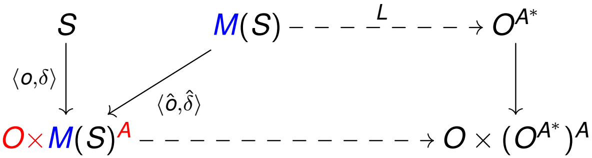

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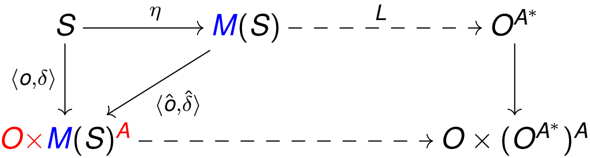

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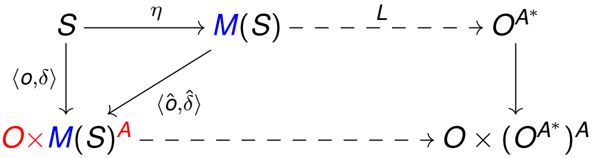

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Now some abstract non-sense

For any monad M and functor F such that FTX is an algebra of the monad M (or: F as a lifting to Set^M) and F has a final coalgebra, we can generalize the construction and semantics:

$$
\begin{array}{ccccc}
S & \xrightarrow{\ \eta\ } & M(S) & \overset{L}{\dashrightarrow} & \Omega \\
{\scriptstyle \langle o,\delta \rangle}\downarrow & \swarrow {\scriptstyle \langle \hat{o},\hat{\delta} \rangle} & & & \downarrow \\
FM(S) & & \dashrightarrow & & F\Omega
\end{array}
$$

This gives rise to determinization constructions for many transitions systems: Mealy machines, structured Moore automata, Pushdown automata, . . .

Conclusions

What I hope you take home. . .

- Coalgebra is not only about semantics but also about algorithms
- Coalgebra is about unifying and generalizing