An algebra for Kripke polynomial coalgebras

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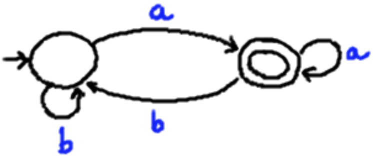
Motivation

Deterministic automata (DA)

- Widely used model in Computer Science.
- Acceptors of languages



Regular expressions

- *User-friendly* alternative to DA notation.
- Many applications: pattern matching (`grep`), specification of circuits, ...

$$b^*a(b^*a)^*$$

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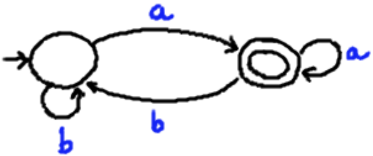

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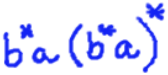

Kleene's Theorem

Let $A \subseteq \Sigma^*$. The following are equivalent.

1. $A = L(\mathcal{A})$, for some finite automaton $\mathcal{A}$.
2. $A = L(r)$, for some regular expression r.

Motivation

Kleene Algebras

- Kleene asked for a complete set of axioms which would allow derivation of all equations among regular expressions.
- Kozen showed that the axioms of Kleene algebras solve this problem.

Axioms

$$
\begin{array}{rcl}
E_1 + E_2 & = & E_2 + E_1 \\
E_1 + (E_2 + E_3) & = & (E_1 + E_2) + E_3 \\
E_1 + E_1 & = & E_1 \\
E + \emptyset & = & E \\
 & \vdots & \\
1 + aa^* & \leq & a^* \\
ax \leq x \rightarrow a^*x & \leq & x
\end{array}
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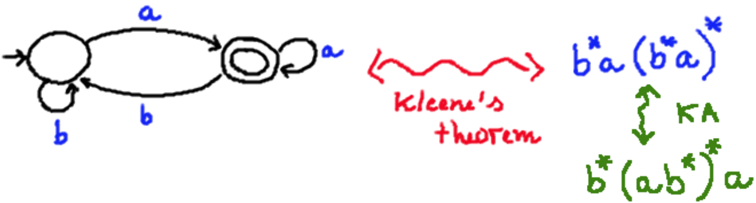

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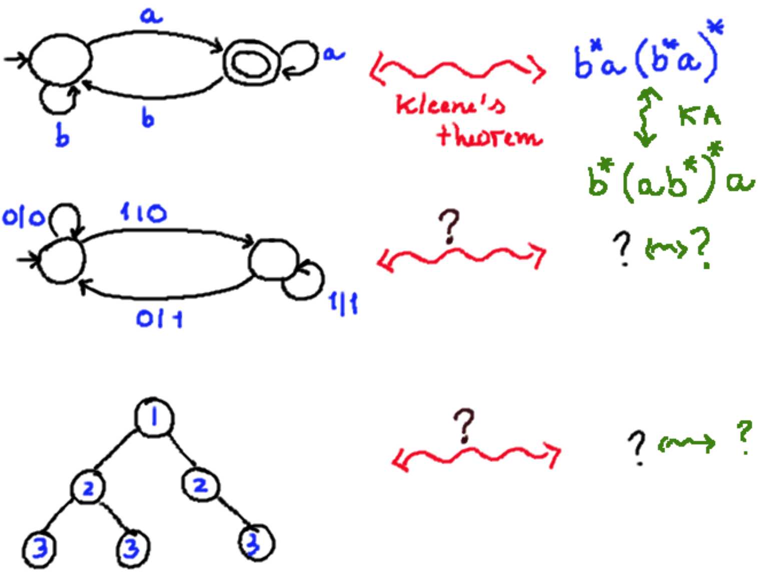

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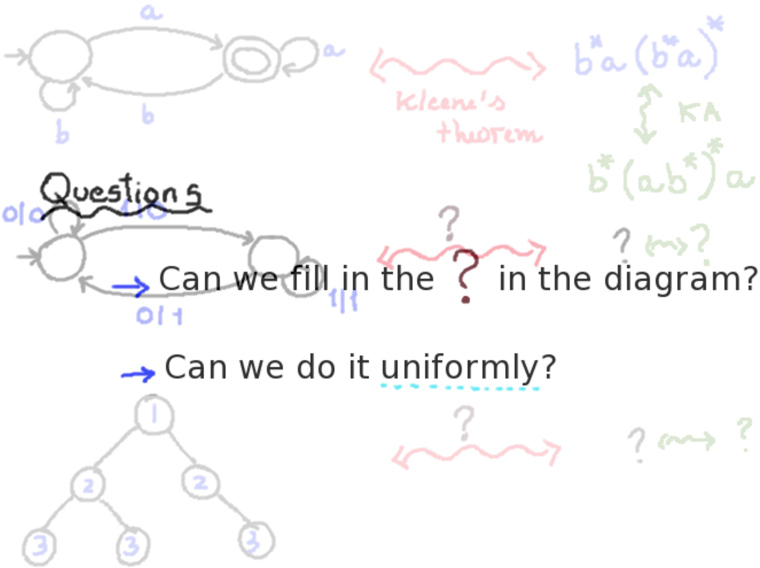

Questions

- Can we fill in the ? in the diagram?
- Can we do it uniformly?

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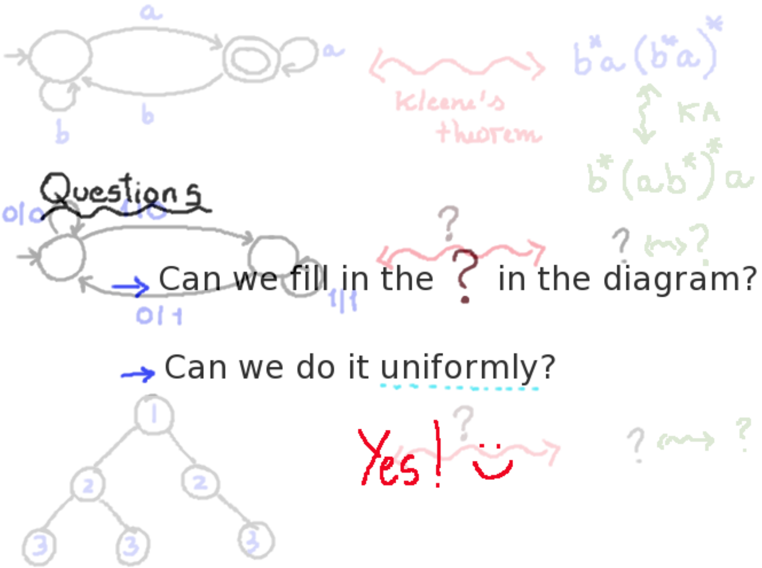

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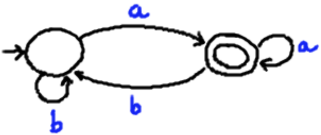
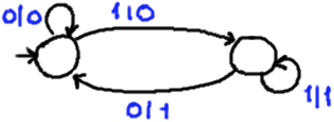
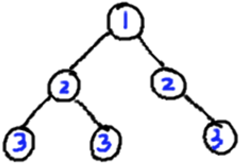
What do these things have in common?



$(S, \delta : S \to 2 \times S^A)$



$(S, \delta : S \to (B \times S)^A)$



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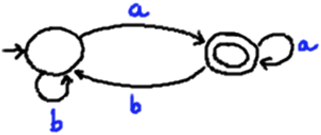

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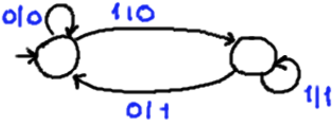

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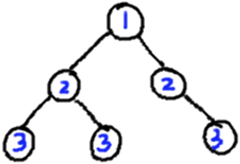

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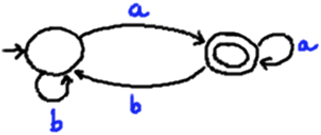

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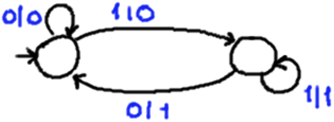

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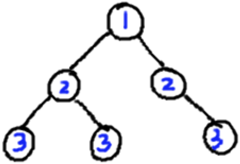

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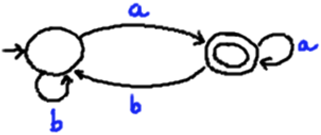

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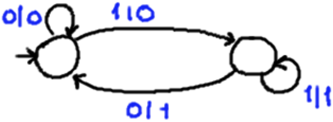

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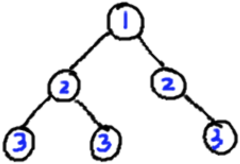

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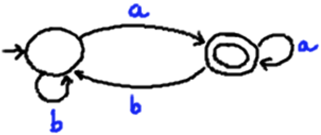

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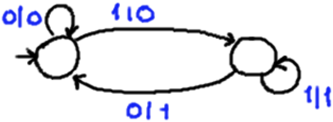

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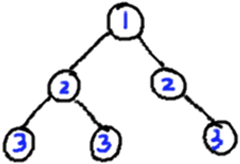

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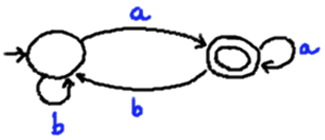

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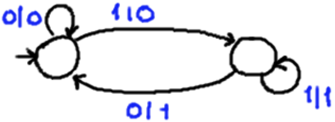

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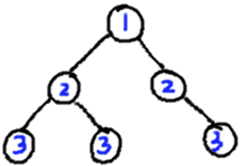

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$(S, \delta : S \to GS)$

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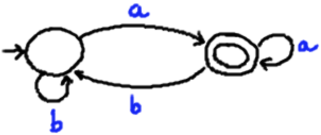

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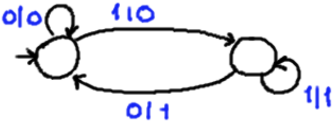

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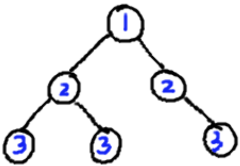

$(S, \delta : S \to (1 + S) \times A \times (1 + S))$

$(S, \delta : S \to GS)$ G**-coalgebras**

Coalgebras

Kripke polynomial coalgebras

- Generalizations of deterministic automata
- Kripke polynomial coalgebras: set of states S and $t : S \to GS$

$$G ::= Id \mid B \mid G \times G \mid G + G \mid G^A \mid \mathcal{P}G$$

Examples

$G = 2 \times Id^A$	Deterministic automata
$G = (B \times Id)^A$	Mealy machines
$G = 1 + (\mathcal{P}Id)^A$	LTS (with explicit termination)
$G = B \times Id^{At \times \Sigma}$	Automata on guarded strings
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Deterministic automata
$Q \to 2 \times Q^{\Sigma}$

$\Updownarrow$

Regular Expressions
Kleene algebra

$\Updownarrow$

Formal Languages

Beyond deterministic automata

Deterministic automata $\rightsquigarrow$ G-coalgebras

$Q \rightarrow 2 \times Q^{\Sigma}$ $\qquad$ $Q \rightarrow GQ$

$\Updownarrow$

Regular Expressions
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Formal Languages

Beyond deterministic automata

Deterministic automata $Q \to 2 \times Q^{\Sigma}$	$\rightsquigarrow$	G-coalgebras $Q \to GQ$
$\Updownarrow$		
Regular Expressions Kleene algebra	$\rightsquigarrow$	?
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Beyond deterministic automata

Deterministic automata $Q \to 2 \times Q^{\Sigma}$	$\rightsquigarrow$	G-coalgebras $Q \to GQ$
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Regular Expressions Kleene algebra	$\rightsquigarrow$	G-expressions G-axioms
$\Updownarrow$		$\Updownarrow$
Formal Languages	$\rightsquigarrow$	Final coalgebra

In a nutshell

Our contributions are:

- A (syntactic) notion of *G-expressions* for polynomial coalgebras: each expression will denote an element of the final coalgebra.
- We show the equivalence between *G*-expressions and finite *G*-coalgebras (analogously to Kleene's theorem).
- For each *G*, we provide a sound and complete equational system for *G*-expressions.

G-expressions

$$E \quad ::= \quad \emptyset \mid \epsilon \mid E \cdot E \mid E + E \mid E^*$$

$$E_G \quad ::= \quad ?$$

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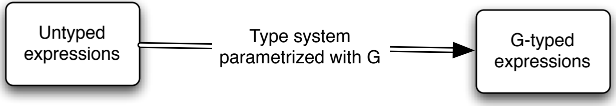
How do we define E_G?



G-expressions

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Exp \ni \varepsilon & ::= & \emptyset \mid \varepsilon \oplus \varepsilon & \mid \mu x.\gamma \\
& & & \mid b \qquad B \\
& & & \mid l\langle\varepsilon\rangle \mid r\langle\varepsilon\rangle \qquad G_1 \times G_2 \\
& & & \mid l[\varepsilon] \mid r[\varepsilon] \qquad G_1 + G_2 \\
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Examples

Deterministic automata expressions – $G = 2 \times Id^A$

$$\varepsilon \quad ::= \quad \emptyset \mid \varepsilon \oplus \varepsilon \mid \mu x.\gamma \mid l\langle 1 \rangle \mid l\langle 0 \rangle \mid r\langle a(\varepsilon) \rangle$$

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Mealy expressions – $G = (B \times Id)^A$

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LTS expressions – $G = 1 + (\mathcal{P}Id)^A$

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Kleene's theorem

The goal is:

G − expressions **correspond to** Finite *G − coalgebras* and vice-versa.

What does it mean **correspond**?

Final coalgebras exist for polynomial coalgebras.

$$\begin{array}{ccccc} S & \overset{h}{\dashrightarrow} & \Omega_G & \overset{[\![\cdot]\!]}{\dashleftarrow} & Exp_G \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \omega_G} & & \\ GS & \underset{Gh}{\dashrightarrow} & G\Omega_G & & \end{array}$$

correspond $\equiv$ **mapped to the same element of the final coalgebra**
$\equiv$ **bisimilar**

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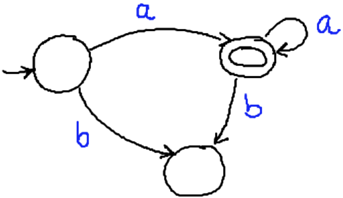
Theorem

1. *Let (S,g) be a G-coalgebra. If S is* finite *then there exists for any $s \in S$ a G-expression ε_s such that $\varepsilon_s \sim s$.*
2. *For all G-expressions ε, there exists a finite G-coalgebra (S, g) such that $\exists_{s \in S}\ s \sim \varepsilon$.*

Examples of application



$$\varepsilon \quad = \quad \mu x.a(1 \oplus x)$$

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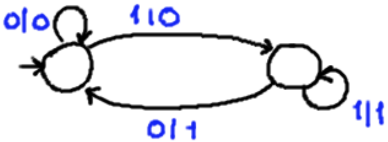
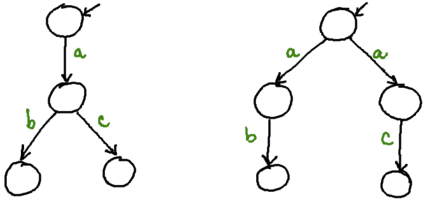


$$\begin{array}{rcl} \varepsilon & = & \mu x.\, 0(x) \oplus 1(\varepsilon') \oplus 0{\downarrow}0 \oplus 1{\downarrow}0 \\ \varepsilon' & = & \mu y.\, 0(x) \oplus 1(y) \oplus 0{\downarrow}1 \oplus 1{\downarrow}1 \end{array}$$

Examples of application



$$\varepsilon_1 = a.(b.\delta \oplus c.\delta) \qquad \varepsilon_2 = a.b.\delta \oplus a.c.\delta$$

Axiomatization

$$\left.\begin{array}{lcl} \varepsilon_1 \oplus \varepsilon_2 & = & \varepsilon_2 \oplus \varepsilon_1 \\ \varepsilon_1 \oplus (\varepsilon_2 \oplus \varepsilon_3) & = & (\varepsilon_1 \oplus \varepsilon_2) \oplus \varepsilon_3 \\ \varepsilon_1 \oplus \varepsilon_1 & = & \varepsilon_1 \\ \varepsilon \oplus \emptyset & = & \varepsilon \end{array}\right\} G$$

$$\left.\begin{array}{lcl} \mu x.\gamma & = & \gamma[\mu x.\gamma/x] \\ \gamma[\varepsilon/x] \leq \varepsilon & \Rightarrow & \mu x.\gamma \leq \varepsilon \end{array}\right\} FP$$

$$\left.\begin{array}{lcl} \emptyset & = & \perp_B \\ b_1 \oplus b_2 & = & b_1 \vee b_2 \end{array}\right\} B$$

Sound and complete w.r.t $\sim$

$$\left.\begin{array}{lcl} l(\emptyset) & = & \emptyset \\ l(\varepsilon_1) \oplus l(\varepsilon_2) & = & l(\varepsilon_1 \oplus \varepsilon_2) \\ r(\emptyset) & = & \emptyset \\ r(\varepsilon_1) \oplus r(\varepsilon_2) & = & r(\varepsilon_1 \oplus \varepsilon_2) \end{array}\right\} G_1 \times G_2$$

Similar for $G_1 + G_2$ and G^A

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Axiomatization – example

LTS expressions – $G = 1 + (\mathcal{P} Id)^A$

$$\varepsilon ::= \emptyset \mid \varepsilon \oplus \varepsilon \mid \mu x.\gamma \mid \underbrace{\surd}_{l[*]} \mid \underbrace{\delta}_{r[\emptyset]} \mid \underbrace{a.\varepsilon}_{r[a(\{\varepsilon\})]}$$

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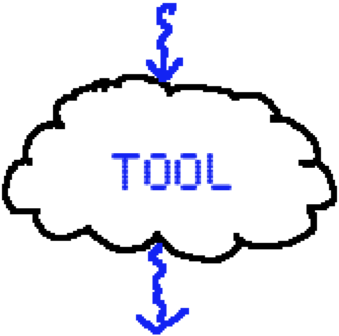
Ultimate goal

Functor G



Language + Axiomatization

Conclusions and Future work

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- Language of regular expressions for polynomial coalgebras
- Generalization of Kleene theorem and Kleene algebra

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