Deriving syntax and axioms for quantitative regular behaviours

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Motivation

- Kleene's theorem and Kleene algebra
- Extension to Kripke polynomial coalgebras (Mealy machines, Binary trees, LTS, . . .)

Can we take it a step further?

Systems where labels may come from a *quantitative alphabet*: monoid.

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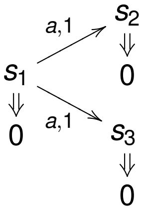
Examples: Weighted automata, probabilistic systems, etc

Weighted automata: a first example



Each **state** has an output value and each transition between two states is assigned a weight.

Weighted automata as coalgebras

Weighted automata as coalgebras

- Output : $S \to \mathbb{M}$
- Transition : $S \to S^A \to \mathbb{M}$
- $S \to \mathbb{M} \times (\mathbb{M}^S)^A$

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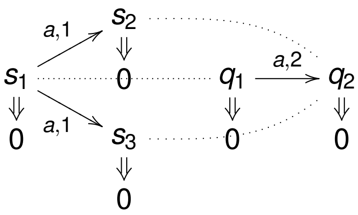
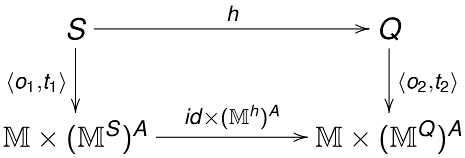
$\mathbb{M}^-$: monoidal exponentiation functor

For a set X: $M^X = \{f \mid f : X \to \mathbb{M},\ f \text{ with finite support}\}$.
For a function $h : X \to Y$, $\mathbb{M}^h$ maps each function $\phi \in \mathbb{M}^X$ to ϕ^h, defined as

$$\phi^h(q) = \sum_{s' \in h^{-1}(q)} \phi(s')$$

Morphism example





For s_1:

$h^{-1}(q_2) = \{s_2, s_3\}$

$t_1(s_1)(a))^h(q_2) =$

$t_1(s_1)(a)(s_2) + t_1(s_1)(a)(s_3) = 2 =$

$t_2(q_1)(a)(q_2) = t_2(h(s_1))(a)(q_2)$

s_2 and s_3: trivial.

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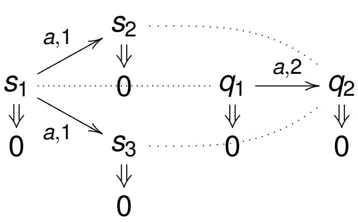

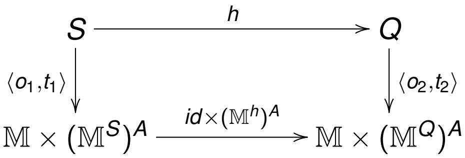

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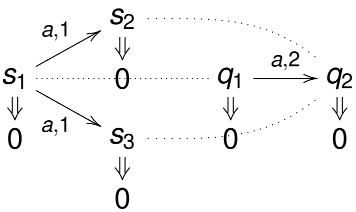

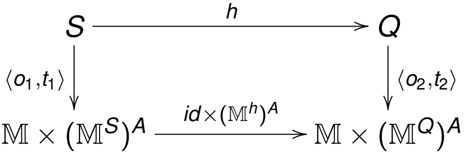

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And now what?

- For polynomial coalgebras

$$G ::= Id \mid B \mid G_1 \times G_2 \mid G_1 + G_2 \mid G^A$$

  we have a generalization of Kleene's theorem and Kleene algebras.
- Can we add $\mathbb{M}^-$ in a similar way? Almost!

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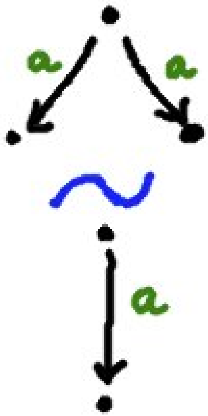
What is different about these *quantitative systems* systems?



Idempotency does not hold anymore: problem for Kleene's theorem and axiomatization.

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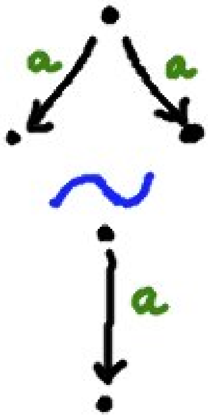

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Recall...

$$
\begin{array}{rcll}
Exp \ni \varepsilon & ::= & \emptyset \mid \varepsilon \oplus \varepsilon \mid \mu x.\gamma & \\
& & \mid b & B \\
& & \mid l\langle\varepsilon\rangle \mid r\langle\varepsilon\rangle & G_1 \times G_2 \\
& & \mid l[\varepsilon] \mid r[\varepsilon] & G_1 + G_2 \\
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Operational semantics: $\lambda_G : Exp_G \to G(Exp_G)$

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Axiomatization:

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\varepsilon_1 \oplus \varepsilon_2 & = & \varepsilon_2 \oplus \varepsilon_1 \\
\varepsilon_1 \oplus (\varepsilon_2 \oplus \varepsilon_3) & = & (\varepsilon_1 \oplus \varepsilon_2) \oplus \varepsilon_3 \\
\varepsilon_1 \oplus \varepsilon_1 & = & \varepsilon_1 \\
\varepsilon \oplus \emptyset & = & \varepsilon
\end{array}\right\} G
$$

$$
\left.\begin{array}{rcl}
\mu x.\gamma & = & \gamma[\mu x.\gamma/x] \\
\gamma[\varepsilon/x] \leq \varepsilon & \Rightarrow & \mu x.\gamma \leq \varepsilon
\end{array}\right\} FP
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$$
\left.\begin{array}{rcl}
\emptyset & = & \bot_B \\
b_1 \oplus b_2 & = & b_1 \vee b_2
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$$
\left.\begin{array}{rcl}
l\langle\emptyset\rangle & = & \emptyset \\
l\langle\varepsilon_1\rangle \oplus l\langle\varepsilon_2\rangle & = & l\langle\varepsilon_1 \oplus \varepsilon_2\rangle \\
r\langle\emptyset\rangle & = & \emptyset \\
r\langle\varepsilon_1\rangle \oplus r\langle\varepsilon_2\rangle & = & r\langle\varepsilon_1 \oplus \varepsilon_2\rangle
\end{array}\right\} G_1 \times G_2
$$

Similar for $G_1 + G_2$ and G^A

Extension to $\mathbb{M}^-$

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\varepsilon_1 \oplus \varepsilon_1 & = & \varepsilon_1, \quad \textit{if } \varepsilon \in Exp_G \\
\varepsilon \oplus \emptyset & = & \varepsilon
\end{array}
$$

$\vdots$

$0 \cdot \varepsilon \equiv \emptyset$

$(m_1 \cdot \varepsilon) \oplus (m_2 \cdot \varepsilon) = (m_1 + m_2) \cdot \varepsilon$

Expressions for weighted automata – $\mathbb{M} \times (\mathbb{M}^{Id})^A$

$\varepsilon ::= \emptyset \mid \varepsilon \oplus \varepsilon \mid \mu x.\varepsilon \mid x \mid m \mid a(m \cdot \varepsilon)$ where $m \in \mathbb{M}$ and $a \in A$

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\begin{array}{l}
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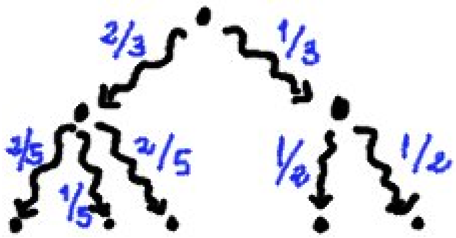
\varepsilon_1 \oplus \varepsilon_2 \equiv \varepsilon_2 \oplus \varepsilon_1 \\
\varepsilon \oplus \emptyset \equiv \varepsilon \\
a(m \cdot \varepsilon) \oplus a(m' \cdot \varepsilon) \equiv a((m + m') \cdot \varepsilon) \\
m \oplus m' \equiv m + m' \\
a(0 \cdot \varepsilon) \equiv \emptyset \qquad 0 \equiv \emptyset \\
\varepsilon[\mu x.\varepsilon / x] \equiv \mu x.\varepsilon \\
\gamma[\varepsilon / x] \equiv \varepsilon \Rightarrow \mu x.\gamma \equiv \varepsilon
\end{array}
$$

Note: s and $a(m \cdot \varepsilon)$ abbreviate $l\langle s \rangle$ and $r\langle a(m \cdot \varepsilon)\rangle$

Probabilistic systems I

Markov chains

- Generalization of labelled transition systems: labels taken from $\mathbb{R}$ and sum equals 1.



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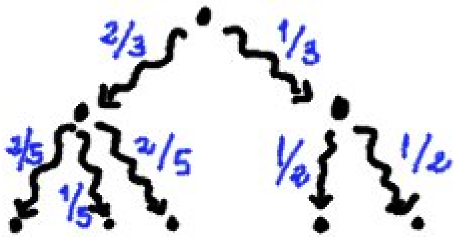
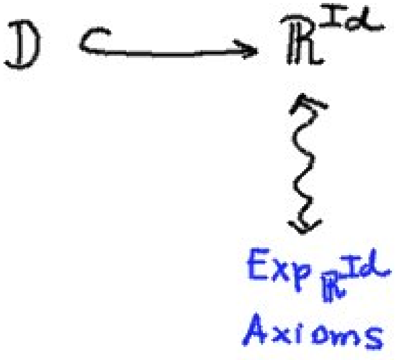


Markov Chains as coalgebras

$D(X) =$ set of probabiliy distributions over X with finite support

D as monoidal exponentiation



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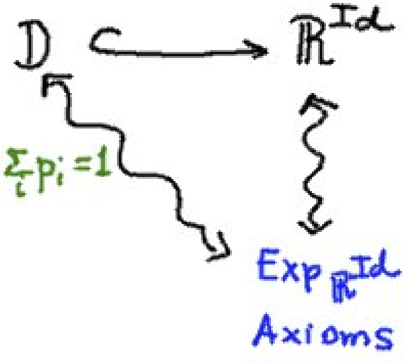

Probabilistic systems II

MC	$\mathcal{D}_\omega$	Markov chains
DA	$(\mathcal{I}d + 1)^A$	deterministic automata
NA	$\mathcal{P}(A \times \mathcal{I}d) \cong \mathcal{P}^A$	non-deterministic automata, LTSs
React	$(\mathcal{D}_\omega + 1)^A$	reactive systems [15,24]
Gen	$\mathcal{D}_\omega(A \times \mathcal{I}d) + 1$	generative systems [24]
Str	$\mathcal{D}_\omega + (A \times \mathcal{I}d) + 1$	stratified systems [24]
Alt	$\mathcal{D}_\omega + \mathcal{P}(A \times \mathcal{I}d)$	alternating systems [8]
Var	$(\mathcal{D}_\omega(A \times \mathcal{I}d) + \mathcal{P}(A \times \mathcal{I}d))/\bowtie$	Vardi systems [25]
SSeg	$\mathcal{P}(A \times \mathcal{D}_\omega)$	simple Segala systems [22,21]
Seg	$\mathcal{P}\mathcal{D}_\omega(A \times \mathcal{I}d)$	Segala systems [22,21]
Bun	$\mathcal{D}_\omega\mathcal{P}(A \times \mathcal{I}d)$	bundle systems [4]
PZ	$\mathcal{P}\mathcal{D}_\omega\mathcal{P}(A \times \mathcal{I}d)$	Pnueli-Zuck systems [18]
MG	$\mathcal{P}\mathcal{D}_\omega\mathcal{P}(A \times \mathcal{I}d + \mathcal{I}d)$	most general systems

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Probabilistic systems III

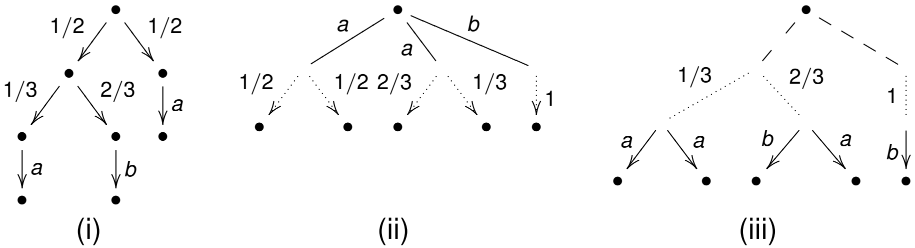

Figure: (i) A stratified system—$D(Id) + (B \times Id) + 1$, (ii) a simple Segala system—$\mathcal{P}(D(Id))^A$ and (iii) a Pnueli-Zuck system—$\mathcal{P}(D(\mathcal{P}Id))^A$

Segala systems — $\mathcal{P}(D_\omega(Id))^A$

Exercise: board. ($\mathcal{P} = 2^-$)

$\varepsilon ::= \emptyset \mid \varepsilon \boxplus \varepsilon \mid \mu x.\varepsilon \mid x \mid a(\{\varepsilon'\})$ where $a \in A$, $p_i \in (0,1]$ and $\sum_{i \in 1\ldots n} p_i = 1$
$\varepsilon' ::= \bigoplus_{i \in 1 \cdots n} p_i \cdot \varepsilon_i$

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$(\varepsilon'_1 \oplus \varepsilon'_2) \oplus \varepsilon'_3 \equiv \varepsilon'_1 \oplus (\varepsilon'_2 \oplus \varepsilon'_3)$ $\quad \varepsilon'_1 \oplus \varepsilon'_2 \equiv \varepsilon'_2 \oplus \varepsilon'_1$
$(p_1 \cdot \varepsilon) \oplus (p_2 \cdot \varepsilon) \equiv (p_1 + p_2) \cdot \varepsilon$
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$(p_1 \cdot \varepsilon) \oplus (p_2 \cdot \varepsilon) \equiv (p_1 + p_2) \cdot \varepsilon$

$\varepsilon[\mu x.\varepsilon / x] \equiv \mu x.\varepsilon$ $\quad \gamma[\varepsilon/x] \equiv \varepsilon \Rightarrow \mu x.\gamma \equiv \varepsilon$

Probabilistic systems – some remarks

- The interplay between non-determinism and probabilities was a serious issue in existing work
- Coalgebraically it is simple and arises from functor composition
- Axiomatizations and expressions existed for some of the systems mentioned but not uniformly.

Conclusions

- Extended our previous framework to quantitative systems.
- Uniform derivation of language and axioms for probabilistic systems: opens the door to new results.
- Not trivial due to lack of idempotency (hidden detail).

Remark: Implementation of this is much harder than for the previous framework, because of extra conditions (e.g. $\sum p_i = 1$).

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