Bi-infinite streams coalgebraically

An (incomplete) exercise in coalgebraic reasoning

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Motivation

- Coinduction is useful for building a calculus for infinite data structures

  [Rutten 2000] Elements of stream calculus (an extensive exercise in coinduction)

  [Silva&Rutten 2007] Behavioural differential equations and coinduction for binary trees.

- We would like to develop a coinductive calculus for bi-infinite streams

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$$\dots\ a_{-3}\ a_{-2}\ a_{-1}\ \underline{a_0}\ a_1\ a_2\ a_3\ \dots$$

Goal of this talk

- Show two different representations for bi-infinite streams
- Hint on how to develop a calculus for this data structure

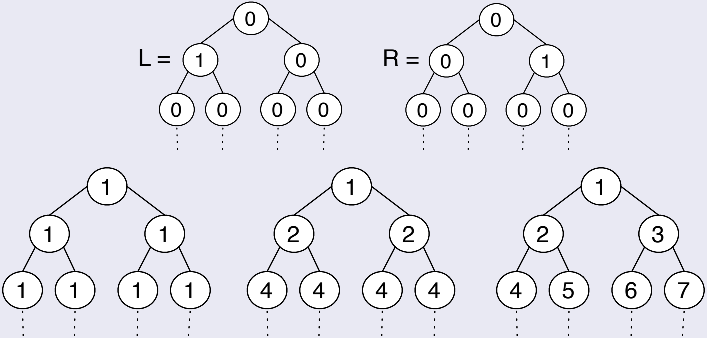
State of the art

Streams [Rutten00]

$X = (0, 1, 0, 0, \ldots)$
$r = (r, 0, 0, 0, \ldots),\ r \in \mathbb{R}$
$(1, 1, 1, 1, \ldots) = \frac{1}{1-X}$
$(1, 2, 4, 8, \ldots) = \frac{1}{1-2X}$
$(1, 2, 3, 4, \ldots) = \frac{1}{(1-X)^2}$

State of the art (cont.)

Infinite binary trees – T_A – [Silva and Rutten 2007]



$\frac{1}{1-L-R}$ $\frac{1}{1-2L-2R}$ $\frac{1}{1-L-R} \times \frac{1-L}{1-2L-2R}$

Bi-infinite streams

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$$A^{\mathbb{Z}} = \{\sigma : \mathbb{Z} \to A\}$$

Questions:

- Is this set the final coalgebra for a given functor F?
- Is there more than one functor?
- If so, how do we choose the best?

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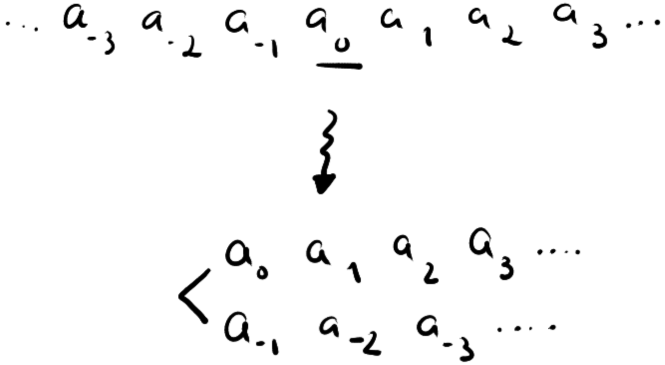
Questions: Short answers

- Is this set the final coalgebra for a given functor F? Yes
- Is there more than one functor? Yes
- If so, how do we choose the best? We don't know (yet)

First Observation



First Observation

$$\ldots a_{-3}\ a_{-2}\ a_{-1}\ \underline{a_0}\ a_1\ a_2\ a_3 \ldots \qquad A^{\mathbb{Z}}$$

$$\rightsquigarrow \qquad\qquad \cong$$

$$\left\langle \begin{matrix} a_0\ a_1\ a_2\ a_3 \ldots \\ a_{-1}\ a_{-2}\ a_{-3} \ldots \end{matrix} \right. \qquad (A \times A)^{\omega}$$

So. . .

- $A^{\mathbb{Z}}$ is the final coalgebra of $F(X) = (A \times A) \times X$
- We could reuse stream calculus

$X = ((0,0), (1,1), (0,0), (0,0), \ldots)$

$r = (\ldots, (0,0), \underline{(r,0)}, (0,0), \ldots),\ r \in \mathbb{R}$

$(\ldots, 1, \underline{1}, 1, 1, \ldots) = \frac{(1,1)}{(1,1)-X}$

But... Is this the only/best representation? Are we fully benefiting from the structure of bi-infinite streams?

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What else can we do

$$\cdots \quad a_{-2} \; a_{-1} \; \underline{a_0} \; a_1 \; a_2 \cdots$$

–

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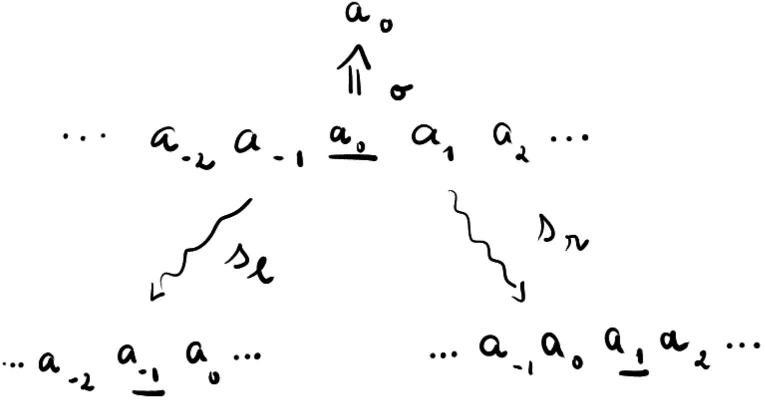

Then...

- $A^{\mathbb{Z}}$ has a dynamics given by:

$$A^{\mathbb{Z}} \xrightarrow{<s_l, o, s_r>} A^{\mathbb{Z}} \times A \times A^{\mathbb{Z}}$$

- Is $A^{\mathbb{Z}}$ the final coalgebra of $G(X) = X \times A \times X$? No
- The final coalgebra of $G(X) = X \times A \times X$ is the set T_A of infinite binary trees
- $A^{\mathbb{Z}}$ is a subcoalgebra of T_A

$$\begin{array}{ccc} A^{\mathbb{Z}} & \xhookrightarrow{\quad h \quad} & T_A \\ \downarrow {\scriptstyle <s_l, o, s_r>} & & \downarrow {\scriptstyle <l, i, r>} \\ A^{\mathbb{Z}} \times A \times A^{\mathbb{Z}} & \longrightarrow & T_A \times A \times T_A \end{array}$$

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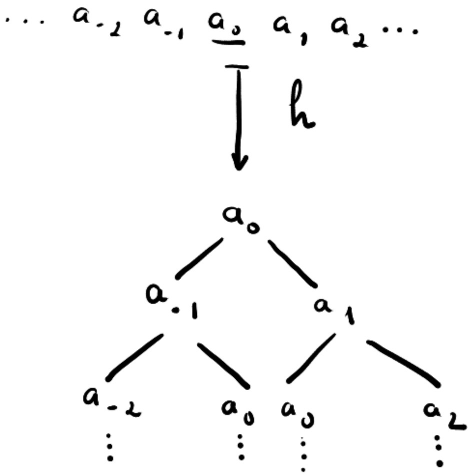
Remarks



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- A tree $\sigma \in T_A$ will be a valid representation of a bi-infinite stream iff

$$l \cdot r(\sigma) \sim \sigma \sim r \cdot l(\sigma)$$

- $A^{\mathbb{Z}} \cong \Box P$

$$P = \{\sigma \in T_A \mid l \cdot r(\sigma) = \sigma = r \cdot l(\sigma)\}$$

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Reusing the tree calculus

$$(\ldots, 1, 1, \underline{1}, 1, 1, \ldots) = \frac{1}{1-L-R}$$

$$(\ldots, 0, 1, \underline{0}, 1, 0, \ldots) = (L+R) \times \frac{1}{1+(L+R)^2}$$

Conclusions and Future work

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- We have shown two possible ways of developing a calculus for bi-infinite streams
- It is not obvious which representation is better

Future work

- Work out more examples in the two representations shown
- Study finite tailed bi-infinite streams (Laurent series) to check if they give rise to a different coalgebra.

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