# (Concrete) Coalgebraic Logics and Synthesis of Mealy Machines

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# Motivation

## Design of Sequential systems

- (Binary) Mealy machines $\leftrightarrow$ digital circuits
- There is no formal way of specifying Mealy machines
- Typically they are "defined" in a natural language, such as English

  **Source of ambiguities**
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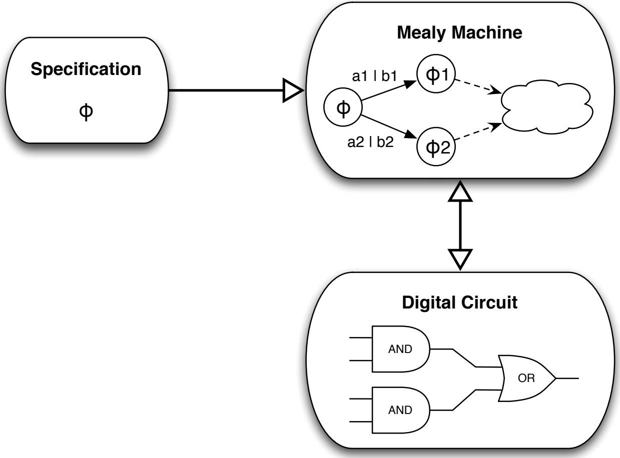

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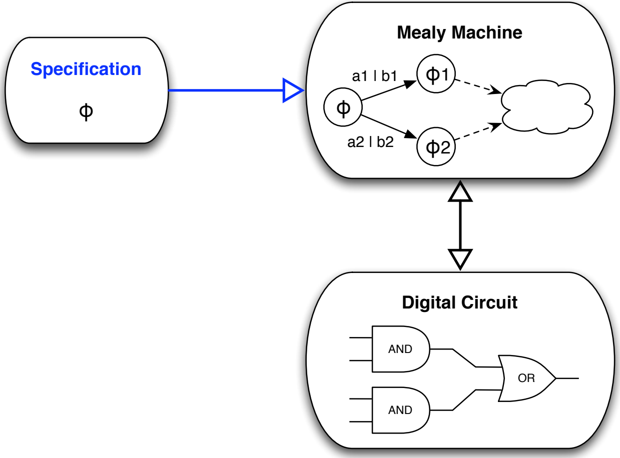

# But first...

What do we mean by **Binary Mealy Machine?**

- Mealy machines = **Deterministic Mealy machines**
- Mealy machine = set of states $S$ + transition function $f$

$$\begin{aligned} f &: S \to (B \times S)^A \\ f(s)(a) &= \; < b, s' > \end{aligned}$$

$$s \xrightarrow{a|b} s'$$

  $A$ is the input alphabet and $B$ is the output alphabet
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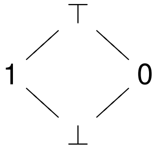
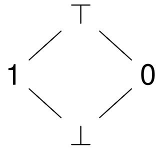
$B = \mathcal{B} =$ 

# Information order

$\mathcal{B} =$




- $\top$ – *abstraction* (under-specification)
- $\bot$ – *inconsistency* (over-specification)
- 0 and 1 – concrete output values.

# Mealy automata are coalgebras

**Observation:**

A Mealy machine is a coalgebra of the functor

$$\begin{aligned} M &: Set \to Set \\ M(X) &= (B \times X)^A \end{aligned}$$

**(Almost) for free:**

- Notion of bisimulation : equivalence between states, minimization
- Specification logic for Mealy machines

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**Remark** Simple but expressive logic: Every finite Mealy machine corresponds to a finite formula in the logic.

# Examples

- Output 0 at each input of 1

$$1 \downarrow 0$$

- Output 0 at each input of two consecutive 1's

$$\nu x.(1(1 \downarrow 0 \wedge 1(x) \wedge 0(x)) \wedge 0(x))$$

- Output 0 at each second input of 1

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# Satisfaction relation

For a Mealy machine $(S, f)$ and $s \in S$:

$$
\begin{array}{lll}
s \models_\eta tt & & \text{for all } s \\
s \models_\eta a(\phi) & \textit{iff} & f(s)(a)_2 \models_\eta \phi \\
s \models_\eta a \downarrow b & \textit{iff} & f(s)(a)_1 \leq_B b \\
s \models_\eta \phi_1 \wedge \phi_2 & \textit{iff} & s \models_\eta \phi_1 \textit{ and } s \models_\eta \phi_2 \\
s \models_\eta x & \textit{iff} & s \in \eta(x) \\
s \models_\eta \nu x.\phi & \textit{iff} & \exists T \subseteq S. s \in T \textit{ and } \forall t \in T. t \models_{\eta[T/x]} \phi
\end{array}
$$

# Satisfaction relation

For a Mealy machine $(S, f)$ and $s \in S$:

$$\begin{array}{lcl} & \vdots & \\ s \models_\eta a(\phi) & \textit{iff} & f(s)(a)_2 \models_\eta \phi \\ s \models_\eta a \downarrow b & \textit{iff} & f(s)(a)_1 = b \\ & \vdots & \end{array}$$

# So far...

- We can specify properties – $\phi$
- Given a mealy machine $(S, f)$ and $s \in S$ we can check if $s \models \phi$
- Next: Given $\phi$ generate $(S, f)$ such that $\exists_{s \in S}\ s \models \phi$.

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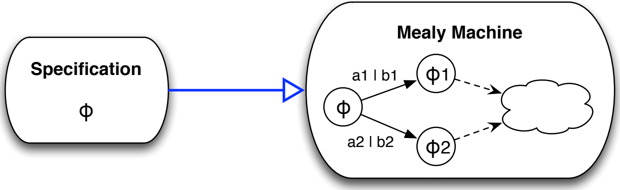
# Synthesis



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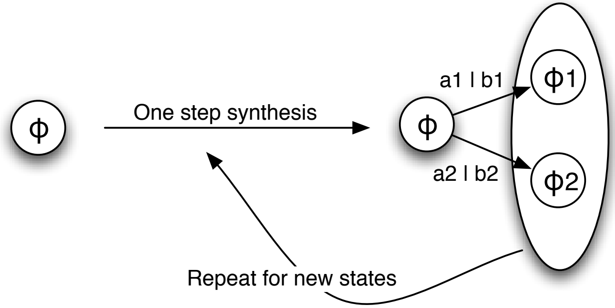
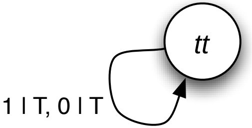


ϕ
One step synthesis
ϕ
a1 l b1
a2 l b2
ϕ1
ϕ2
Repeat for new states

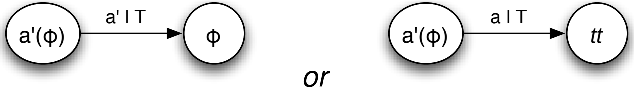
# One-step synthesis

$$\delta(tt)(a) = < \top, tt >$$



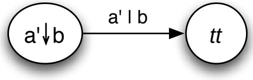
# One-step synthesis

$$\delta(a'(\phi))(a) = \begin{cases} <\top, \phi> & a = a' \\ <\top, tt> & \textit{otherwise} \end{cases}$$

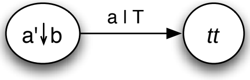


# One-step synthesis

$$\delta(a' \downarrow b)(a) = \begin{cases} < b, tt > & a = a' \\ < \top, tt > & otherwise \end{cases}$$




*or*



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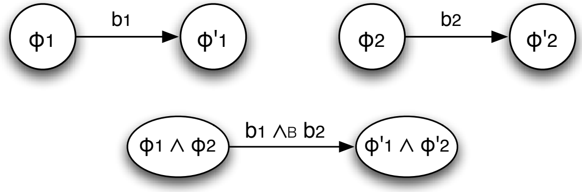

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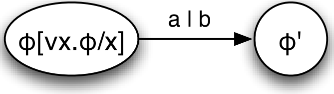

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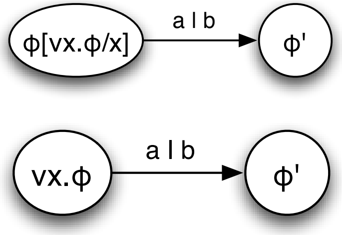

# Examples

- $\phi = 1 \downarrow 0 \wedge (\nu x.1(x))$

$$\begin{aligned} \delta(\phi)(0) &= <\top, tt> \\ \delta(\phi)(1) &= \delta(1 \downarrow 0)(1) \sqcap \delta(\nu x.1(x))(1) \\ &= <0, tt> \sqcap <\top_B, \nu x.1(x)> \\ &= <0, \nu x.1(x)> \end{aligned}$$

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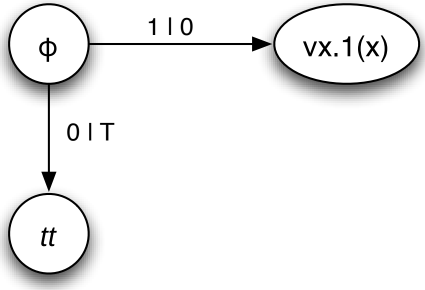

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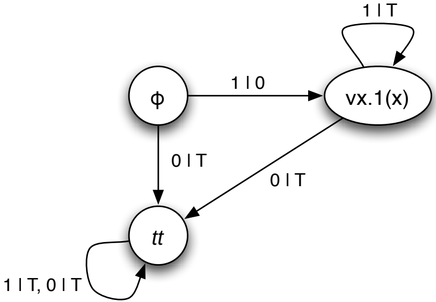

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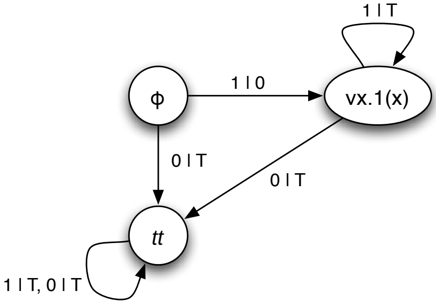


**Remark:** Not minimal.

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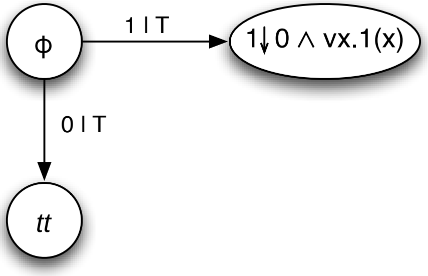

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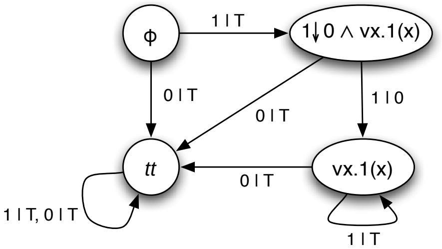

# Conclusions

- Coalgebraic approach : bisimulation and logics
- New logic for Mealy machines
- Synthesis algorithm (implemented in HASKELL)

# Future work

## We did not show...

- Semantics in terms of causal functions
- Proof system (sound and complete)

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- Generalize this approach to other type of automata: Moore automata, weighted automata, ...
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