Behavioural differential equations for binary trees

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Motivation

- Previous work by Jan:

  Behavioural differential equations: a coinductive calculus of streams, automata, and power series

  Elements of stream calculus (an extensive exercise in coinduction)

  showed that coinduction and behavioural differential equations are effective for stream calculus
- We want to investigate if the same approach is effective for other infinite structures, *e.g.* infinite binary trees

What will we show?

We will show how to...

- ... develop a calculus for binary trees *à la formal power series*
- ... define infinite binary trees through *behavioural differential equations*
- ... calculate closed expressions for infinite binary trees

Formal power series

The set of infinite binary trees – T_A – is the final coalgebra of

$$F(X) = X \times A \times X$$

Recall: A formal power series is a function $\sigma : X^* \rightarrow k$ where X is the set of variables (or input symbols) and k is a semiring.

For A semiring, the set T_A is a formal power series over X=2 (**Why?**), *i.e*,

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Why?

- Final coalgebra for $G(X) = A \times X^B$ is A^{B^*}
- $F(X) = X \times A \times X \cong A \times X \times X \cong A \times X^2$
- $2 = \{L, R\}$

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Behavioural Differential Equations

The formal definition of $f(\sigma_1, \ldots, \sigma_n) = \sigma \in T_A$ is now expressed in terms of a *behavioural differential equation*.

$$
\begin{array}{rcll}
\sigma(\varepsilon) & = & c & \textit{initial value} \\
\sigma_L & = & \textit{left\_exp} & \textit{left derivative} \\
\sigma_R & = & \textit{right\_exp} & \textit{right derivative}
\end{array}
$$

- We know that such system has a unique solution if:
  1. c is calculated only involving $\sigma_1(\varepsilon), \ldots, \sigma_n(\varepsilon)$
  2. left_exp only depends on $\sigma_1, \ldots, \sigma_n, (\sigma_1)_L, \ldots, (\sigma_n)_L$ and constants
  3. right_exp only depends on $\sigma_1, \ldots, \sigma_n, (\sigma_1)_R, \ldots, (\sigma_n)_R$ and constants

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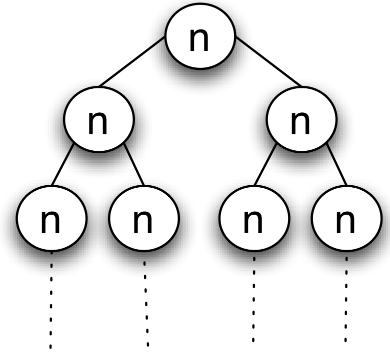
Examples I



$$\sigma(\varepsilon) = n$$
$$\sigma_L = \sigma$$
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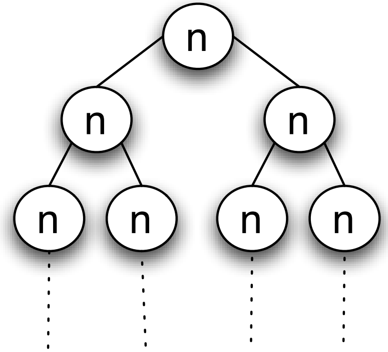

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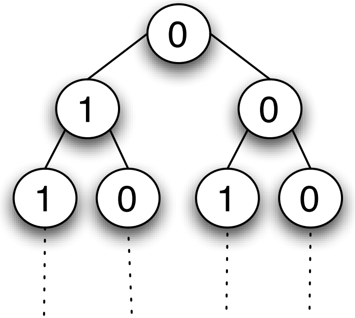
Examples II



$$\begin{aligned} \sigma(\varepsilon) &= 0 \\ \sigma_L &= \sigma + [1] \\ \sigma_R &= \sigma \end{aligned}$$

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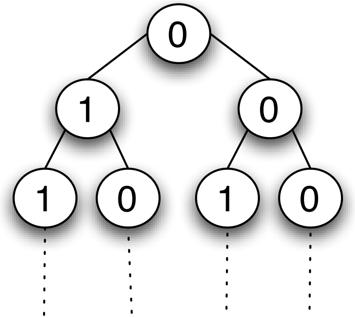

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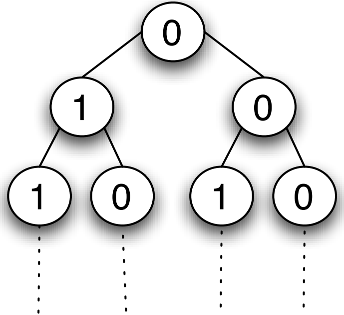

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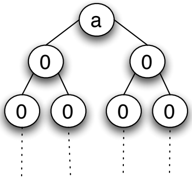
[*a*] denotes



Examples III – The Thue-Morse sequence

- Obtained from the parities of the counts of 1's in the binary representation of non negative integers.
- 0,1,1,0,1,0,0,1, ...
- Can be obtained by the substitution map $\{0 \to 01; 1 \to 10\}$:

$$0 \to 01 \to 0110 \to 01101001 \to \dots$$

- Tree representation (at level k, we have 2^k digits)

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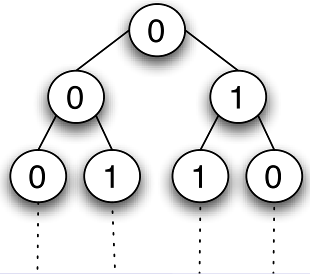
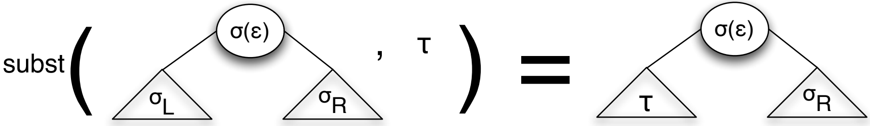


$$\begin{array}{rcl} \sigma(\varepsilon) & = & 0 \\ \sigma_L & = & \sigma \\ \sigma_R & = & \sigma + \mathit{repeat}(1) \end{array}$$

Examples IV – Substitution operation



$$
\begin{aligned}
(subst(\sigma,\tau))(\varepsilon) &= \sigma(\varepsilon)\\
(subst(\sigma,\tau))_L &= \tau\\
(subst(\sigma,\tau))_R &= \sigma_R
\end{aligned}
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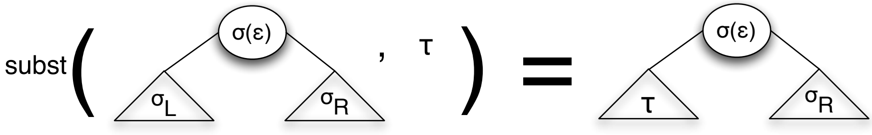

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Operations on trees

From formal power series we inherit several definitions of operations:

Name	Sum	Product
Initial value	$(\sigma+\tau)(\varepsilon)=\sigma(\varepsilon)+\tau(\varepsilon)$	$(\sigma \times \tau)(\varepsilon) = \sigma(\varepsilon) \times \tau(\varepsilon)$
Left der.	$(\sigma + \tau)_L = \sigma_L + \tau_L$	$(\sigma \times \tau)_L = \sigma_L \times \tau + \sigma(\varepsilon) \times \tau_L$
Right der	$(\sigma + \tau)_R = \sigma_R + \tau_R$	$(\sigma \times \tau)_R = \sigma_R \times \tau + \sigma(\varepsilon) \times \tau_R$

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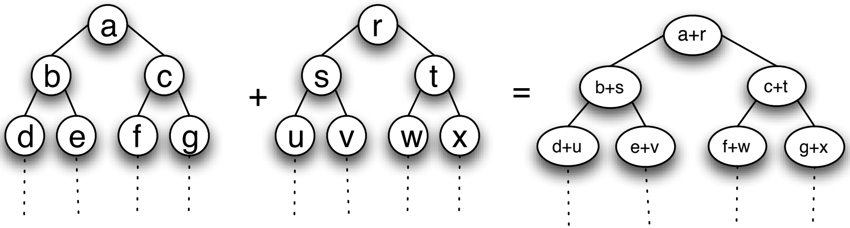

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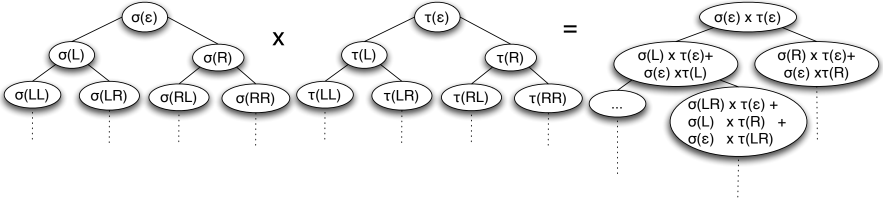

$$(\sigma \times \tau)(w) = \sum_{w=uv} \sigma(u)\tau(v)$$

Fundamental Theorem

For all infinite binary trees $\sigma \in T_A$:

$$\sigma = \sigma(\varepsilon) + L \times \sigma_L + R \times \sigma_R$$

where

$$\begin{array}{cc} L(\varepsilon) = 0 & R(\varepsilon) = 0 \\ L_L = [1] & R_L = [0] \\ L_R = [0] & R_R = [1] \end{array}$$

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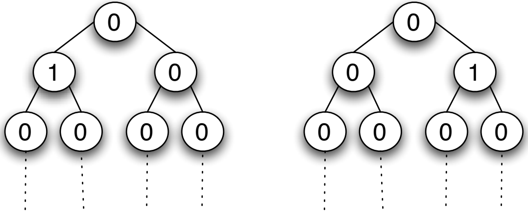
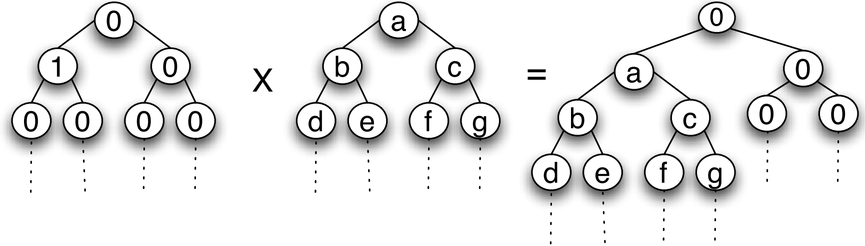


$L \times \sigma_L$



Why?

$$
\begin{aligned}
(L \times \sigma)(\varepsilon) &= L(\varepsilon) \times \sigma(\varepsilon) = 0 \\
(L \times \sigma)_L &= L_L \times \sigma + [L(\varepsilon)] \times \sigma_L \\
&= [1] \times \sigma + [0] \times \sigma_L = \sigma \\
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&= [0] \times \sigma + [0] \times \sigma_R = [0]
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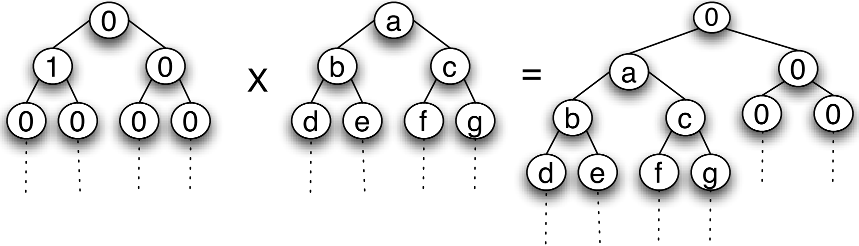

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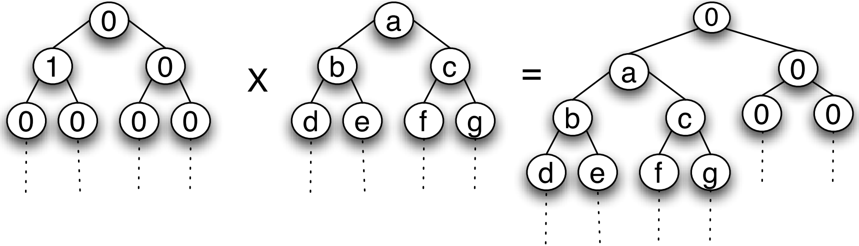

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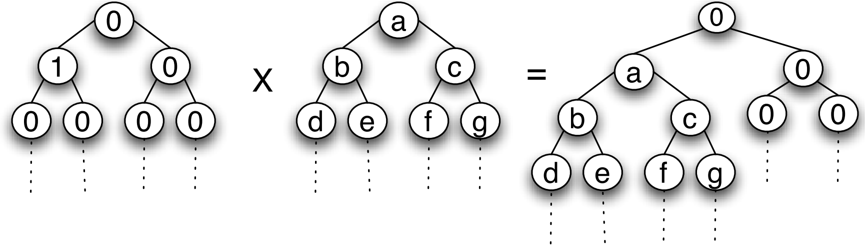

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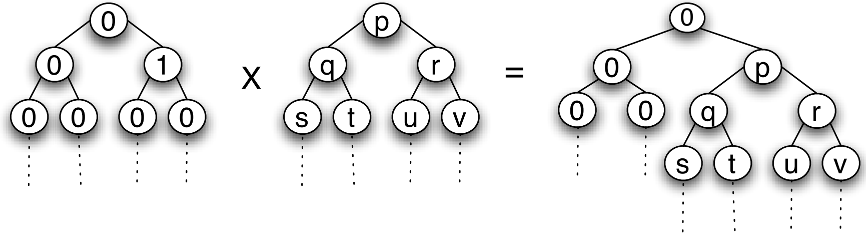
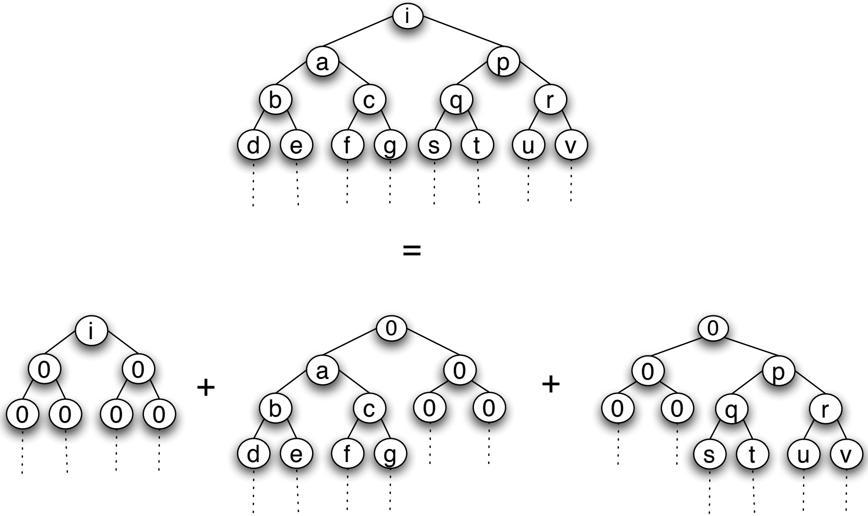
\end{aligned}$$

$R \times \sigma_R$

Similarly:



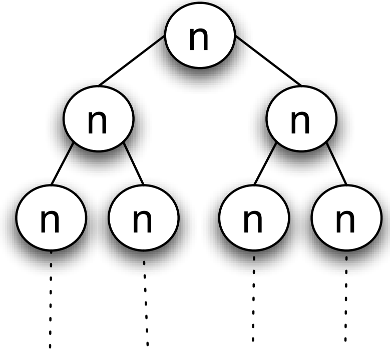
$\sigma = \sigma(\varepsilon) + L \times \sigma_L + R \times \sigma_R$



But. . . What can we do with this theorem?

?

Examples Revisited I



$$\begin{aligned} \sigma(\varepsilon) &= n \\ \sigma_L &= \sigma \\ \sigma_R &= \sigma \end{aligned}$$

Closed Formula I

$$\begin{aligned} & \sigma = [n] + L \times \sigma + R \times \sigma \\ \Leftrightarrow \quad & (1 - L - R)\sigma = [n] \\ \Leftrightarrow \quad & \sigma = \frac{n}{1-L-R} \end{aligned}$$

$\frac{x}{y}$ stands for $y^{-1} \times x$

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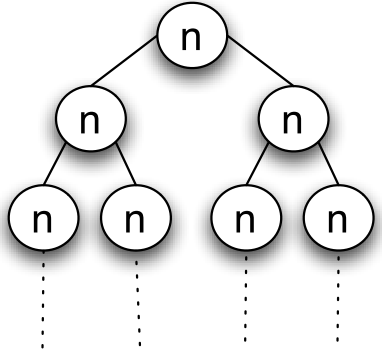

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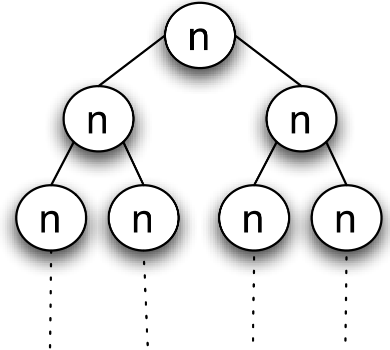

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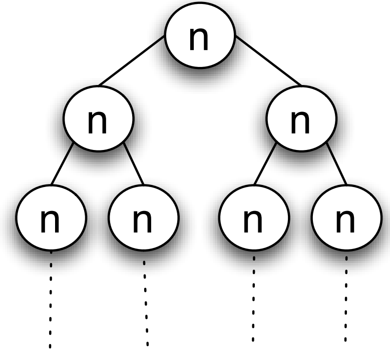

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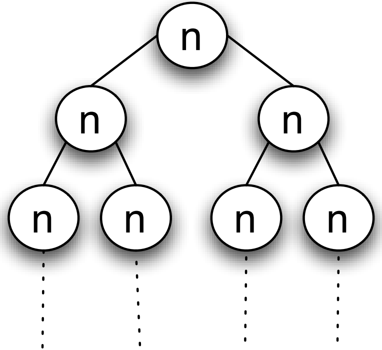

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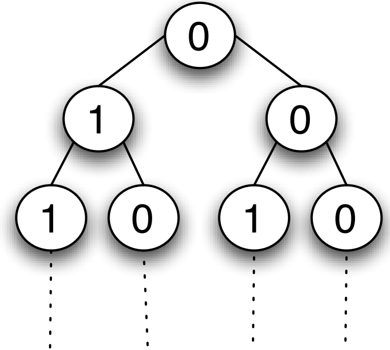
Inverse operation

The inverse of a tree – σ^{-1} – is defined formally so that $\sigma \times \sigma^{-1} = 1$.

$$
\begin{aligned}
\sigma^{-1}(\varepsilon) &= (\sigma(\varepsilon))^{-1} \\
(\sigma^{-1})_L &= (\sigma(\varepsilon))^{-1} \times \sigma_L \times \sigma^{-1} \\
(\sigma^{-1})_R &= (\sigma(\varepsilon))^{-1} \times \sigma_R \times \sigma^{-1}
\end{aligned}
$$

Examples Revisited II

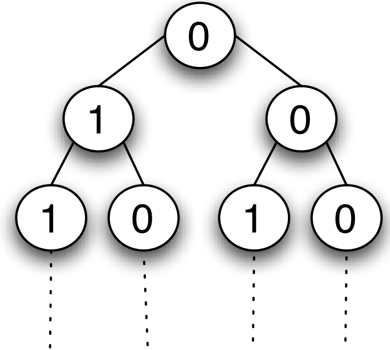


$$\sigma(\varepsilon) = 0$$
$$\sigma_L = \sigma + [1]$$
$$\sigma_R = \sigma$$

Closed Formula II

$$\sigma = 0 + L \times (\sigma + 1) + R \times \sigma$$
$$\Leftrightarrow \quad (1 - L - R)\sigma = L$$
$$\Leftrightarrow \quad \sigma = \frac{L}{1-L-R}$$

Examples Revisited II



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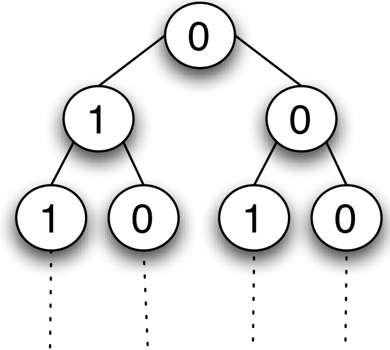

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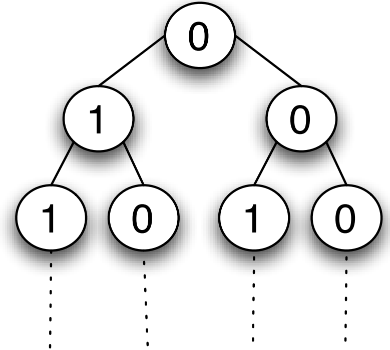

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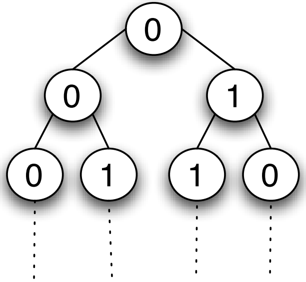
Examples revisited III – The Thue-Morse sequence



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$$\Leftrightarrow \quad (1 - L - R)\sigma = R \times \frac{1}{1-L-R}$$

$$\Leftrightarrow \quad \sigma = \frac{1}{1-L-R} \times R \times \frac{1}{(1-L-R)}$$

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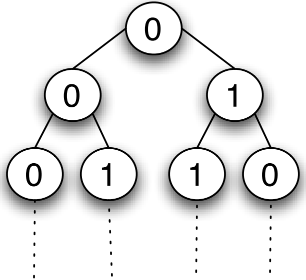

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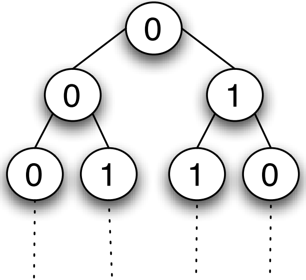
$$\Leftrightarrow \quad \sigma = \frac{1}{1-L-R} \times R \times \frac{1}{(1-L-R)}$$

Examples revisited III – The Thue-Morse sequence

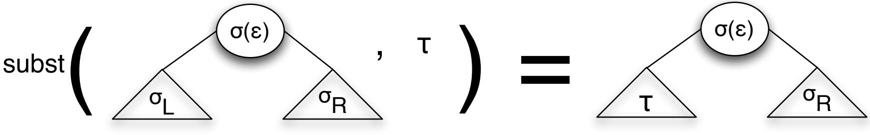


$$\begin{array}{rcl} \sigma(\varepsilon) & = & 0 \\ \sigma_L & = & \sigma \\ \sigma_R & = & \sigma + \mathit{repeat}(1) \end{array}$$

Closed Formula III

$$\begin{array}{rc} & \sigma = 0 + L \times \sigma + R \times (\sigma + \mathit{repeat}(1)) \\ \Leftrightarrow & (1 - L - R)\sigma = R \times \frac{1}{1-L-R} \\ \Leftrightarrow & \sigma = \frac{1}{1-L-R} \times R \times \frac{1}{(1-L-R)} \end{array}$$

Examples revisited IV – Substitution



$$
\begin{aligned}
(subst(\sigma,\tau))(\varepsilon) &= \sigma(\varepsilon)\\
(subst(\sigma,\tau))_L &= \tau\\
(subst(\sigma,\tau))_R &= \sigma_R
\end{aligned}
$$

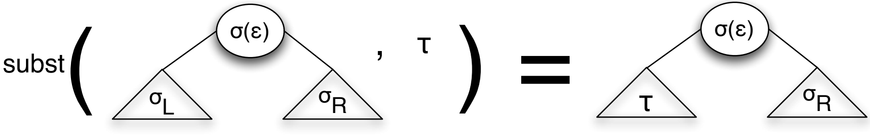
Closed Formula IV

$$subst(\sigma,\tau) = \sigma(\varepsilon) + L \times \tau + R \times \sigma_R$$

$$\Leftrightarrow \quad \{\sigma - L \times \sigma_L = \sigma(\varepsilon) + R \times \sigma_R\}$$

$$subst(\sigma,\tau) = \sigma - L \times (\sigma_L - \tau)$$

Examples revisited IV – Substitution

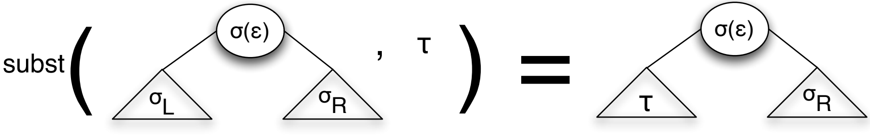


$$(subst(\sigma, \tau))(\varepsilon) = \sigma(\varepsilon)$$
$$(subst(\sigma, \tau))_L = \tau$$
$$(subst(\sigma, \tau))_R = \sigma_R$$

Closed Formula IV

$$subst(\sigma, \tau) = \sigma(\varepsilon) + L \times \tau + R \times \sigma_R$$
$$\Leftrightarrow \quad \{\sigma - L \times \sigma_L = \sigma(\varepsilon) + R \times \sigma_R\}$$
$$subst(\sigma, \tau) = \sigma - L \times (\sigma_L - \tau)$$

Examples revisited IV – Substitution



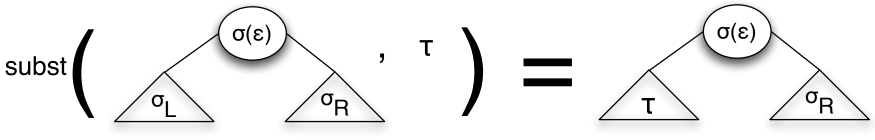
$$
\begin{aligned}
(\mathit{subst}(\sigma,\tau))(\varepsilon) &= \sigma(\varepsilon)\\
(\mathit{subst}(\sigma,\tau))_L &= \tau\\
(\mathit{subst}(\sigma,\tau))_R &= \sigma_R
\end{aligned}
$$

Closed Formula IV

$$\mathit{subst}(\sigma,\tau) = \sigma(\varepsilon) + L \times \tau + R \times \sigma_R$$

$$\Leftrightarrow \quad \{\, \sigma - L \times \sigma_L = \sigma(\varepsilon) + R \times \sigma_R \,\}$$

$$\mathit{subst}(\sigma,\tau) = \sigma - L \times (\sigma_L - \tau)$$



$$\begin{aligned}(\mathit{subst}(\sigma,\tau))(\varepsilon) &= \sigma(\varepsilon)\\ (\mathit{subst}(\sigma,\tau))_L &= \tau\\ (\mathit{subst}(\sigma,\tau))_R &= \sigma_R\end{aligned}$$

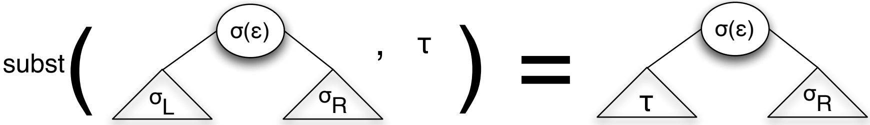
Closed Formula IV

$$\mathit{subst}(\sigma,\tau) = \sigma(\varepsilon) + L \times \tau + R \times \sigma_R$$

$$\Leftrightarrow \qquad \{\, \sigma - L \times \sigma_L = \sigma(\varepsilon) + R \times \sigma_R \,\}$$

$$\mathit{subst}(\sigma,\tau) = \sigma - L \times (\sigma_L - \tau)$$

Examples revisited IV – Substitution



$$\begin{aligned}(subst(\sigma,\tau))(\varepsilon) &= \sigma(\varepsilon)\\ (subst(\sigma,\tau))_L &= \tau\\ (subst(\sigma,\tau))_R &= \sigma_R\end{aligned}$$

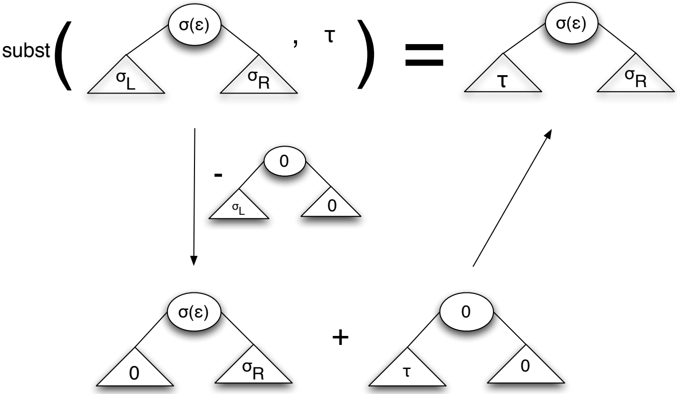
Closed Formula IV

$$subst(\sigma,\tau) = \sigma(\varepsilon) + L \times \tau + R \times \sigma_R$$

$$\Leftrightarrow \qquad \{\, \sigma - L \times \sigma_L = \sigma(\varepsilon) + R \times \sigma_R \,\}$$

$$subst(\sigma,\tau) = \sigma - L \times (\sigma_L - \tau)$$

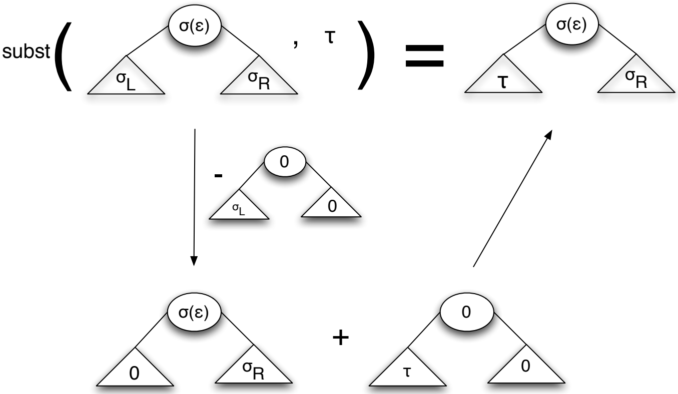
$subst(\sigma, \tau) = \sigma - L \times (\sigma_L - \tau)$



Easily generalizes:

$subst(\sigma, \tau, Path) = \sigma - Path(\sigma_{Path} - \tau)$

$subst(\sigma, \tau) = \sigma - L \times (\sigma_L - \tau)$



Easily generalizes:

$subst(\sigma, \tau, Path) = \sigma - Path(\sigma_{Path} - \tau)$

Conclusions

- Behavioural differential equations are effective to represent infinite binary trees.
- Closed expressions constitute a compact representation of trees (only involving constants) and…
- Give a recipe to implement algorithms.

Future work

- Behavioural differential equations are closely related to lazy functional programming implementations.
- In particular, we would like to study the relation between closed expressions and elimination of corecursion
- We would also like to understand better the class of rational trees