A Kleene theorem for Kripke polynomial coalgebras

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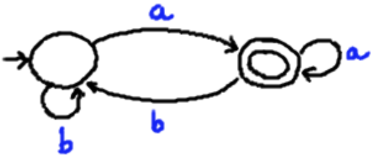
Coalgebra day, March 2009

Motivation

Deterministic automata (DA)

- Widely used model in Computer Science.
- Acceptors of languages



Regular expressions

- *User-friendly* alternative to DA notation.
- Many applications: pattern matching (`grep`), specification of circuits, ...

$$b^*a(b^*a)^*$$

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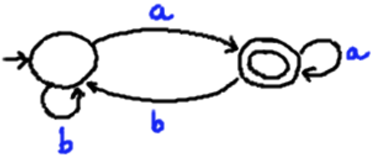

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Kleene's Theorem

Let $A \subseteq \Sigma^*$. The following are equivalent.

1. $A = L(\mathcal{A})$, for some finite automaton $\mathcal{A}$.
2. $A = L(r)$, for some regular expression r.

Motivation

Kleene Algebras

- Kleene asked for a complete set of axioms which would allow derivation of all equations among regular expressions.
- Kozen showed that the axioms of Kleene algebras solve this problem.

Axioms

$$
\begin{array}{rcl}
E_1 + E_2 & = & E_2 + E_1 \\
E_1 + (E_2 + E_3) & = & (E_1 + E_2) + E_3 \\
E_1 + E_1 & = & E_1 \\
E + \emptyset & = & E \\
 & \vdots & \\
1 + aa^* & \leq & a^* \\
ax \leq x \to a^* x & \leq & x
\end{array}
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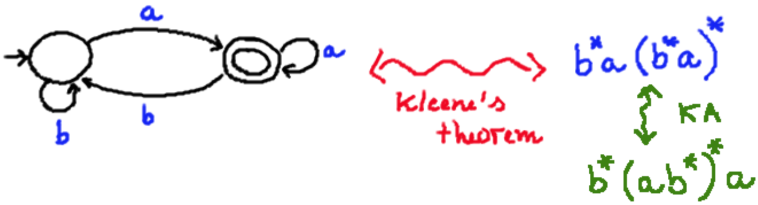

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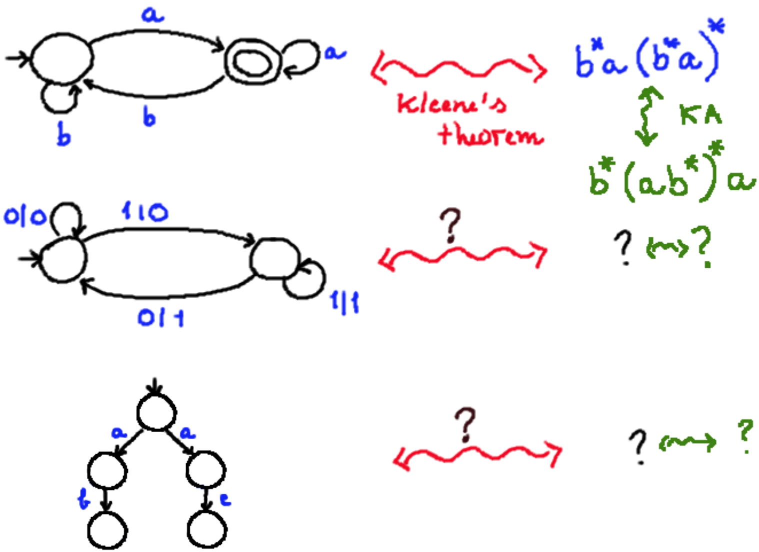

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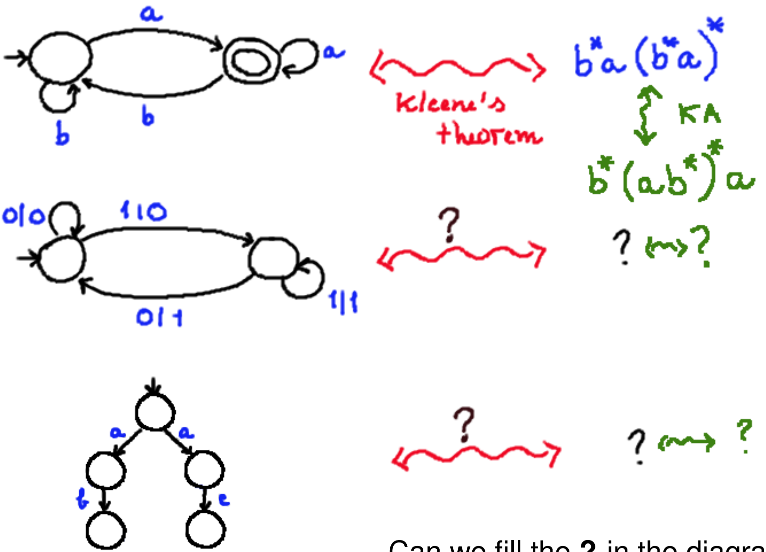
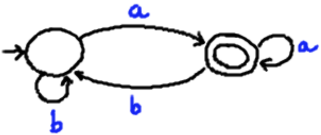
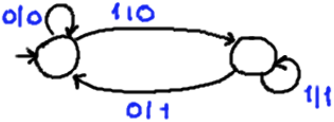


Can we fill the **?** in the diagram?

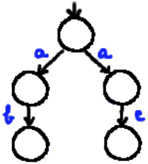
What do these things have in common?



$(S, \delta : S \to 2 \times S^A)$



$(S, \delta : S \to (B \times S)^A)$



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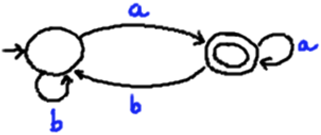

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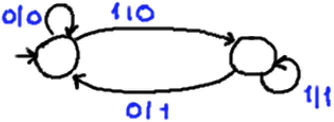

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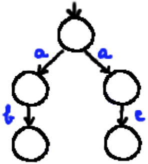

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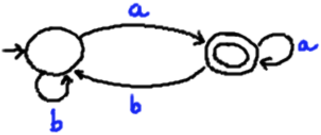

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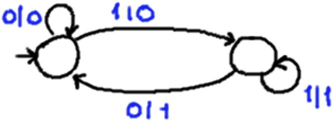

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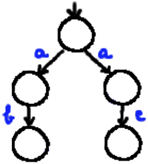

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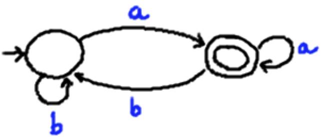

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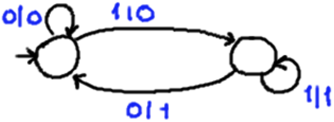

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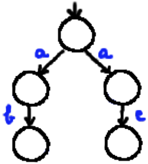

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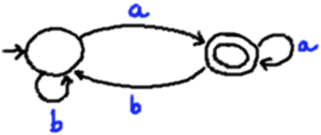

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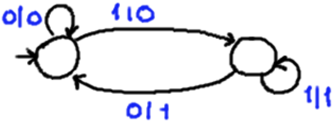

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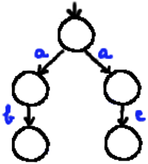

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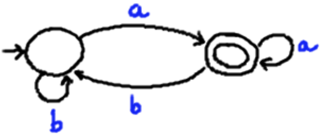

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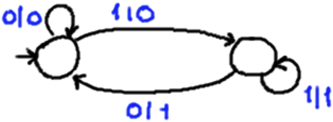

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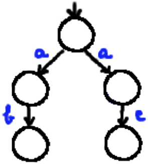

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$(S, \delta : S \to GS)$

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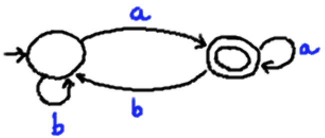

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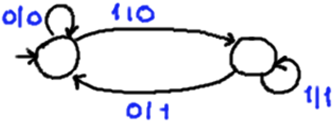

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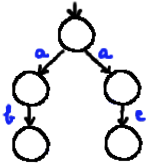

$$(S, \delta : S \to 1 + (\mathcal{P}S)^A)$$

$(S, \delta : S \to GS)$ G**-coalgebras**

Coalgebras

Kripke polynomial coalgebras

- Generalizations of deterministic automata
- Kripke polynomial coalgebras: set of states S and $t : S \to GS$

$$G ::= Id \mid B \mid G \times G \mid G + G \mid G^A \mid \mathcal{P}G$$

$\mathcal{P}$ finite

Examples

- $G = 2 \times Id^A$ Deterministic automata
- $G = (B \times Id)^A$ Mealy machines
- $G = 1 + (\mathcal{P}Id)^A$ LTS (with explicit termination)
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In a nutshell — beyond deterministic automata

Deterministic automata $Q \to 2 \times Q^{\Sigma}$	$\rightsquigarrow$	G-coalgebras $Q \to GQ$
$\Updownarrow$		$\Updownarrow$
Regular Expressions	$\rightsquigarrow$	G-expressions
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Formal Languages	$\rightsquigarrow$	Final coalgebra

Our contributions are:

- A (syntactic) notion of *G-expressions* for polynomial coalgebras: each expression will denote an element of the final coalgebra.
- Equivalence between G-expressions and finite G-coalgebras (analogously to Kleene's theorem).
- Sound and complete equational system for G-expressions.

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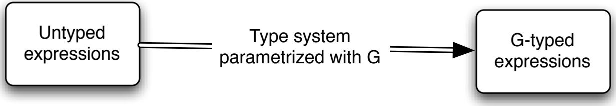
How do we define E_G?



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Examples

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Kleene's theorem

The goal is:

$G-$*expressions* **correspond to** Finite $G-$*coalgebras* and vice-versa.
What does it mean **correspond**?

Final coalgebras exist for Kripke polynomial coalgebras.

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correspond $\equiv$ **mapped to the same element of the final coalgebra**
$\equiv$ **bisimilar**

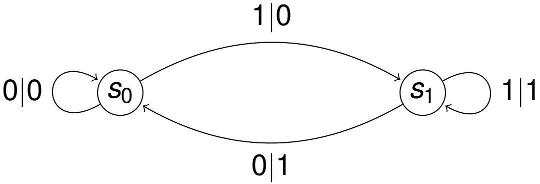
A generalized Kleene theorem

G-coalgebras $\Leftrightarrow$ G-expressions

Theorem

1. *Let (S,g) be a G-coalgebra. If S is* finite *then there exists for any* $s \in S$ *a G-expression* ε_s *such that* $\varepsilon_s \sim s$.
2. *For all G-expressions* ε, *there exists a finite G-coalgebra* (S, g) *such that* $\exists_{s \in S}\ s \sim \varepsilon$.

Proof by example I



$$x_0 = 0(x_0) \oplus 0{\downarrow}0 \oplus 1(x_1) \oplus 1{\downarrow}0$$
$$x_1 = 0(x_0) \oplus 0{\downarrow}1 \oplus 1(x_1) \oplus 1{\downarrow}1$$

Solve the system and take the *least* solution:

$$\varepsilon_0 = \mu x_0.0(x_0) \oplus 0{\downarrow}0 \oplus 1(\varepsilon_1) \oplus 1{\downarrow}0$$
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$\varepsilon_0 \sim s_0$ and $\varepsilon_1 \sim s_1$

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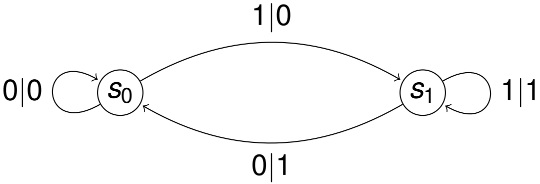

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$$\varepsilon_0 = \mu x_0.0(x_0) \oplus 0{\downarrow}0 \oplus 1(\varepsilon_1) \oplus 1{\downarrow}0$$
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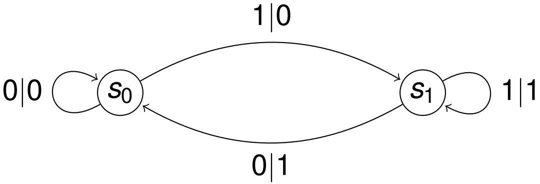
$\varepsilon_0 \sim s_0$ and $\varepsilon_1 \sim s_1$

Proof by example I



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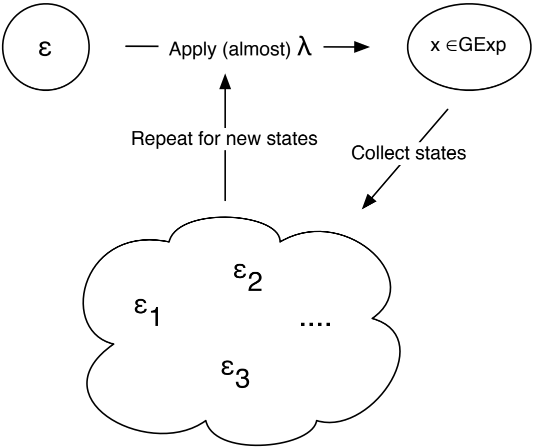
Proof by example II



Proof by example II

$$\varepsilon = \mu x.r\langle a(r\langle b(x)\rangle)\rangle \oplus l\langle 1\rangle$$

$$\varepsilon \xrightarrow{\lambda_a} \langle 1, r\langle b(\varepsilon)\rangle\rangle \xrightarrow{\lambda_b} \langle 1, \varepsilon\rangle$$

Proof by example II

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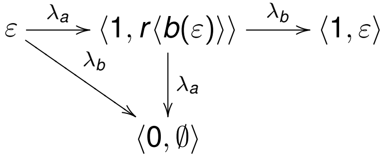

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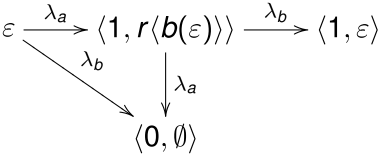

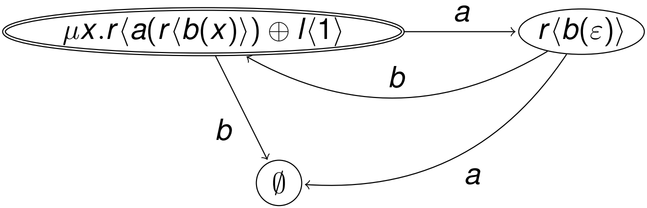

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We need **ACI**!

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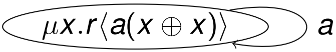

Axiomatization

$$\left.\begin{array}{lcl} \varepsilon_1 \oplus \varepsilon_2 & = & \varepsilon_2 \oplus \varepsilon_1 \\ \varepsilon_1 \oplus (\varepsilon_2 \oplus \varepsilon_3) & = & (\varepsilon_1 \oplus \varepsilon_2) \oplus \varepsilon_3 \\ \varepsilon_1 \oplus \varepsilon_1 & = & \varepsilon_1 \\ \varepsilon \oplus \emptyset & = & \varepsilon \end{array}\right\} G$$

$$\left.\begin{array}{lcl} \mu x.\gamma & = & \gamma[\mu x.\gamma/x] \\ \gamma[\varepsilon/x] \leq \varepsilon & \Rightarrow & \mu x.\gamma \leq \varepsilon \end{array}\right\} FP$$

$$\left.\begin{array}{lcl} \emptyset & = & \perp_B \\ b_1 \oplus b_2 & = & b_1 \vee b_2 \end{array}\right\} B$$

Sound and complete w.r.t $\sim$

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Similar for $G_1 + G_2$ and G^A

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Axiomatization – example

LTS expressions – $G = 1 + (\mathcal{P} Id)^A$

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\varepsilon \oplus \emptyset & = & \varepsilon \\
\varepsilon \oplus \delta & = & \varepsilon
\end{array}$$

No rule

$a.(\varepsilon_1 \oplus \varepsilon_2) = a.\varepsilon_1 \oplus a.\varepsilon_2$

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