Deriving syntax and axioms for quantitative regular behaviours

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Motivation

Specify and reason about systems.

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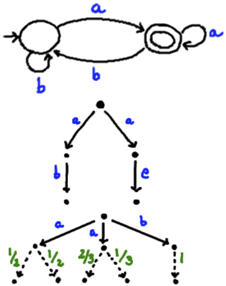
state-machines
e.g. DFA, LTS, PA,



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Specify and **reason** about **systems**.

Syntax
RE, CCS, ...

$b^*a(b^*a)^*$

$a.b.0 + a.c.0$

$a.(\frac{1}{2} \cdot 0 \oplus \frac{1}{2} \cdot 0) + \cdots$

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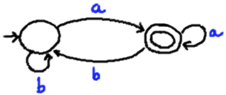

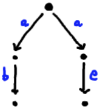

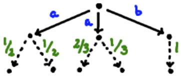

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Axiomatization
KA,. . .

$1 + a\,a^* = a^*$

$\vdots$

$P + 0 = P$

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$p.P \oplus p'.P = (p + p').P$

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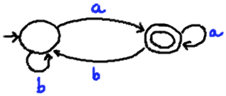

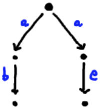

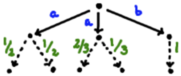

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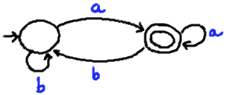
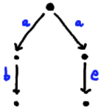
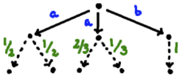
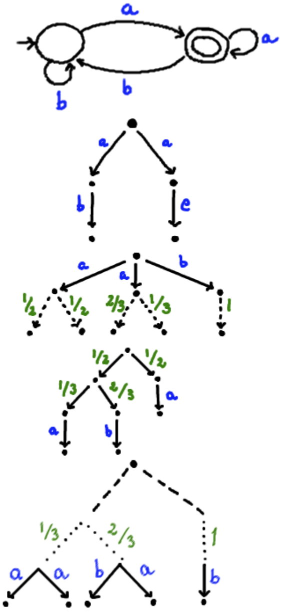






Can we do all of this uniformly in a single framework?

What do this things have in common?



$(S, t{:}S \to 2 \times S^A)$

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$(S, t{:}S \to GS)$ G-coalgebras

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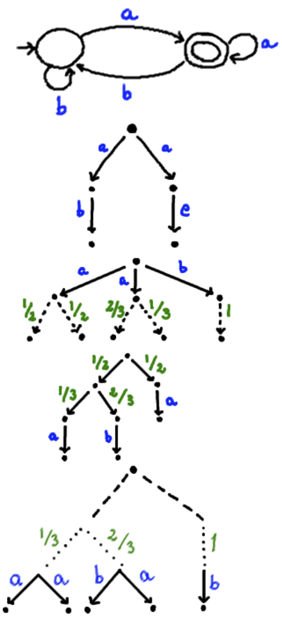

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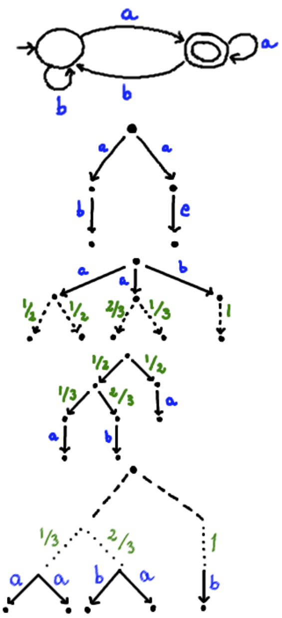

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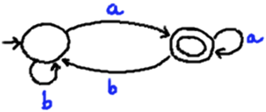

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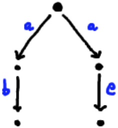

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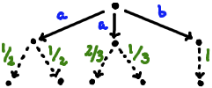

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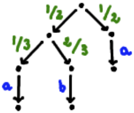

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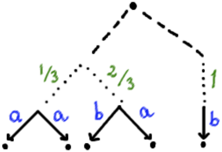

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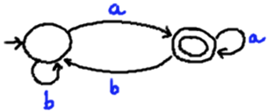

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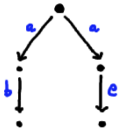

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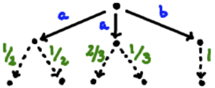

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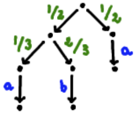

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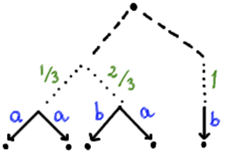

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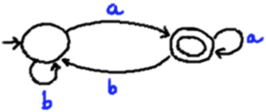

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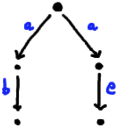

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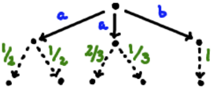

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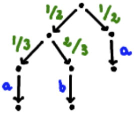

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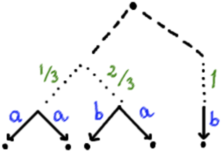

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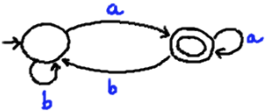

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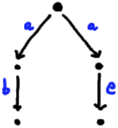

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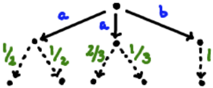

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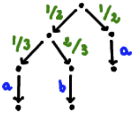

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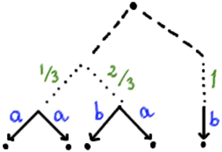

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The power of G

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The functor G determines:

1. notion of observational equivalence (coalg. bisimulation)
2. behaviour (final coalgebra)
3. set of expressions describing finite systems
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1 + 2 are classic coalgebra; 3 + 4 are LICS'09 and CONCUR'09

Coalgebras

Quantitative coalgebras

- Generalizations of deterministic automata
- Quantitative coalgebras: set of states S and $t : S \to GS$

$$G ::= Id \mid B \mid G \times G \mid G + G \mid G^A \mid \mathbb{M}^G$$

$\mathbb{M}$ is a monoid. $\mathcal{P} = 2^{Id}$ and $\mathcal{D}_\omega = \mathbb{R}^{Id}$

Examples

- $G = 2 \times Id^A$ Deterministic automata
- $G = (B \times Id)^A$ Mealy machines
- $G = (\mathcal{P} Id)^A$ LTS
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In this talk. . .

- . . . we present a systematic way to derive from the functor G: languages of (generalized) regular expressions and
- . . . sound and complete axiomatizations thereof for quantitative systems;
- . . . we show the correspondence between language and systems (generalizing Kleene's theorem);
- . . . we apply the framework to several types of probabilistic automata recovering old results and deriving new ones.

G-expressions

$$E \quad ::= \quad \emptyset \mid \epsilon \mid E \cdot E \mid E + E \mid E^*$$

$$E_G \quad ::= \quad ?$$

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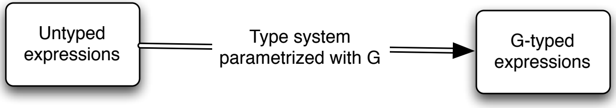
How do we define E_G?



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Exp \ni \varepsilon & ::= & \emptyset \mid \varepsilon \oplus \varepsilon & \mid \mu x.\gamma & \\
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Markov Chain expressions – $G = \mathcal{D}_\omega(Id) = \mathbb{R}^{Id}$

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Markov Chain expressions – $G = \mathcal{D}_\omega(Id) = \mathbb{R}^{Id}$ $\sum_i p_i = 1$

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Examples

LTS expressions – $G = (\mathcal{P} Id)^A = (2^{Id})^A$

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Kleene's Theorem

The goal is:

$G-$ *expressions* **correspond to** Finite $G-$ *coalgebras* and vice-versa.
What does it mean **correspond**?

Final coalgebras exist for quantitative coalgebras.

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correspond $\equiv$ **mapped to the same element of the final coalgebra**
$\equiv$ **bisimilar**

A generalized Kleene Theorem

Theorem

1. *Let (S,g) be a G-coalgebra. If S is* finite *then there exists for any* $s \in S$ *a G-expression* ε_s *such that* $\varepsilon_s \sim s$.
2. *For all G-expressions* ε, *there exists a finite G-coalgebra* (S, g) *such that* $\exists_{s \in S}\ s \sim \varepsilon$.

The proof provides algorithms to construct an expression from a system and vice-versa.

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Axiomatization

$$\left.\begin{array}{lcl} \varepsilon_1 \oplus \varepsilon_2 & \equiv & \varepsilon_2 \oplus \varepsilon_1 \\ \varepsilon_1 \oplus (\varepsilon_2 \oplus \varepsilon_3) & \equiv & (\varepsilon_1 \oplus \varepsilon_2) \oplus \varepsilon_3 \\ \varepsilon_1 \oplus \varepsilon_1 & \equiv & \varepsilon_1, \quad G \text{ polynomial} \\ \varepsilon \oplus \emptyset & \equiv & \varepsilon \end{array}\right\} G$$

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Similar for $G_1 + G_2$ and G^A

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Same syntax and axioms as in [Deng and Palamidessi'05]

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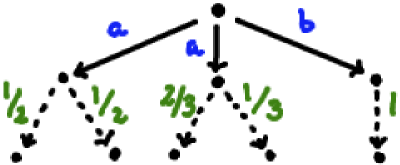

$\Uparrow$

Kleene's Theorem

$\Downarrow$

$$a(\{1/2 \cdot \emptyset \oplus 1/2 \cdot \emptyset\}) \boxplus a(\{1/3 \cdot \emptyset \oplus 2/3 \cdot \emptyset\}) \boxplus b(\{1 \cdot \emptyset\})$$

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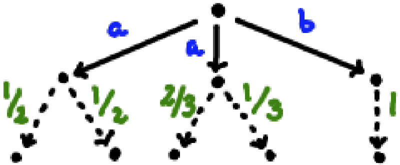

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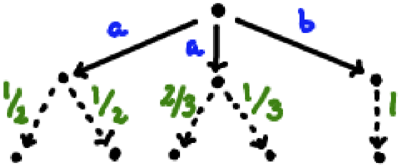

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Results II : Stratified systems – $D_\omega(Id) + (B \times Id) + 1$

$$\varepsilon ::= \mu x.\varepsilon \mid x \mid \langle b, \varepsilon\rangle \mid \bigoplus_{i\in 1\cdots n} p_i \cdot \varepsilon_i \mid \downarrow$$

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New syntax and axiomatization.

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New syntax and axiomatization.

Conclusions and future work

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- Framework to uniformly derive language and axioms for quantitative coalgebras (weighted automata, probabilistic automata, etc)
- Examples show the effectiveness of the framework: known syntaxes recovered, new ones derived.

Future work

- Apply the framework to other systems, *e.g.* alternating systems.
- Automation: `Circ` — Coinductive prover