A Kleene theorem for Polynomial coalgebras

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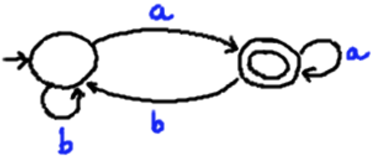
Motivation

Deterministic automata (DA)

- Widely used model in Computer Science.
- Acceptors of languages



Regular expressions

- *User-friendly* alternative to DA notation.
- Many applications: pattern matching (`grep`), specification of circuits, ...

$$b^*a(b^*a)^*$$

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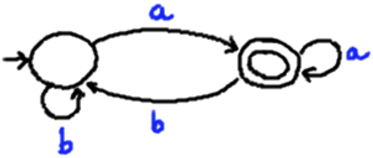

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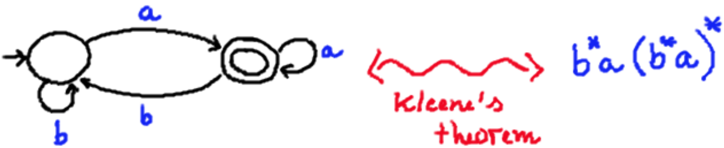
Kleene's Theorem

Let $A \subseteq \Sigma^*$. The following are equivalent.

1. $A = L(\mathcal{A})$, for some finite automaton $\mathcal{A}$.
2. $A = L(r)$, for some regular expression r.

Motivation



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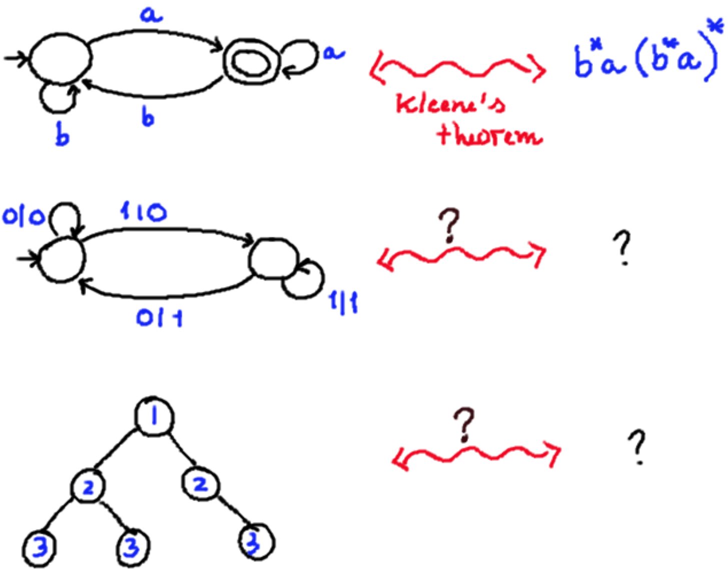

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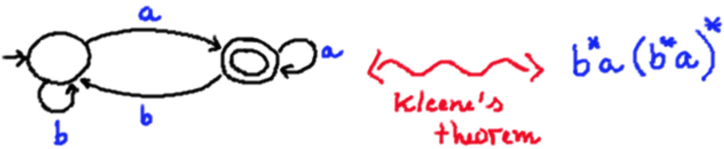
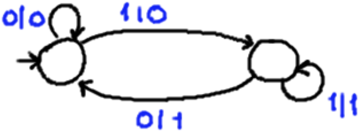
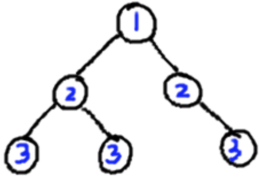
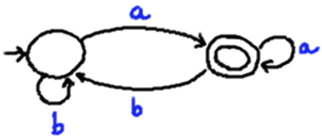




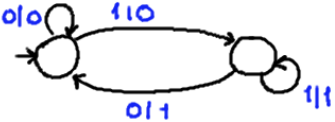


Can we fill the **?** in the diagram?

What do these things have in common?



$(S, \delta : S \to 2 \times S^A)$



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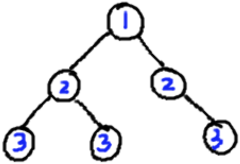

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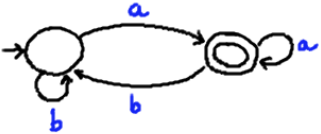

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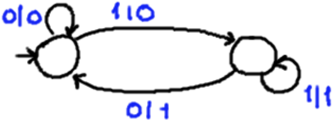

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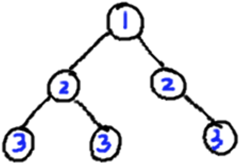

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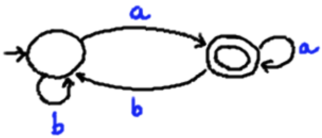

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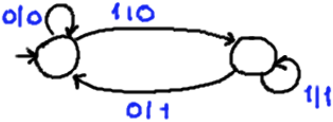

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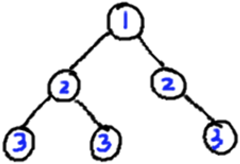

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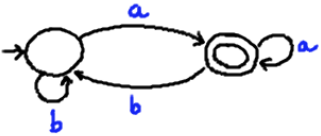

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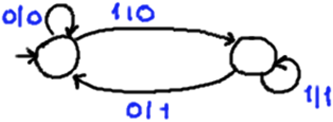

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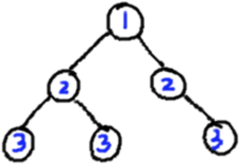

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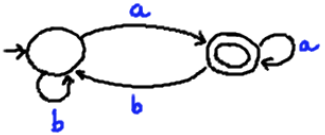

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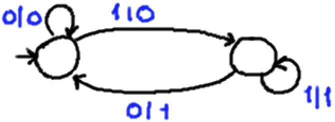

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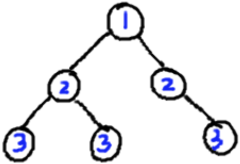

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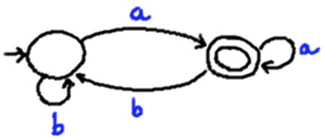

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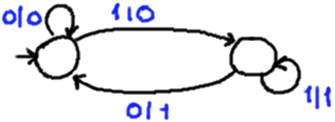

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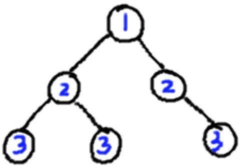

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$$(S, \delta : S \to GS)$$

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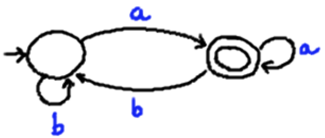

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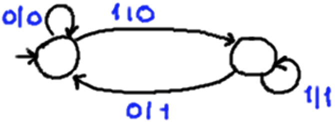

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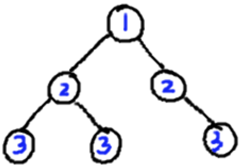

$(S, \delta : S \to (1 + S) \times A \times (1 + S))$

$(S, \delta : S \to GS)$ G-**coalgebras**

Coalgebras

Polynomial coalgebras

- Generalizations of deterministic automata
- Polynomial coalgebras: set of states S and $t : S \to GS$

$$G ::= Id \mid B \mid G \times G \mid G + G \mid G^A$$

Examples

- $G = 2 \times Id^A$ Deterministic automata
- $G = (B \times Id)^A$ Mealy machines
- $G = (1 + Id) \times A \times (1 + Id)$ Binary trees
- ...

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In a nutshell — beyond deterministic automata

Deterministic automata $Q \to 2 \times Q^\Sigma$	$\rightsquigarrow$	G-coalgebras $Q \to GQ$
$\Updownarrow$		$\Updownarrow$
Regular Expressions	$\rightsquigarrow$	G-expressions
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Formal Languages	$\rightsquigarrow$	Final coalgebra

Our contributions are:

- A (syntactic) notion of *G-expressions* for polynomial coalgebras: each expression will denote an element of the final coalgebra.
- Equivalence between G-expressions and finite G-coalgebras (analogously to Kleene's theorem).

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E & ::= & \emptyset \mid \epsilon \mid E \cdot E \mid E + E \mid E^* \\
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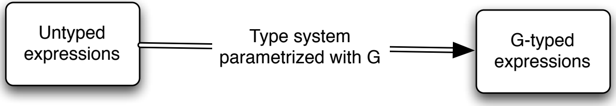
How do we define E_G?



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Mealy expressions – $G = (B \times Id)^A$

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Binary tree expressions – $G = (1 + Id) \times A \times (1 + Id)$

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Kleene's theorem

The goal is:

$G - $ *expressions* **correspond to** Finite $G - $ *coalgebras* and vice-versa.
What does it mean **correspond**?

Final coalgebras exist for Kripke polynomial coalgebras.

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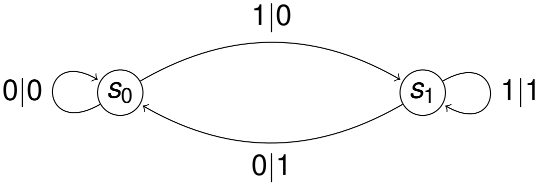
correspond $\equiv$ **mapped to the same element of the final coalgebra**
$\equiv$ **bisimilar**

Theorem

1. *Let (S,g) be a G-coalgebra. If S is* finite *then there exists for any* $s \in S$ *a G-expression* ε_s *such that* $\varepsilon_s \sim s$.
2. *For all G-expressions* ε*, there exists a finite G-coalgebra* (S, g) *such that* $\exists_{s \in S}\ s \sim \varepsilon$.

Proof by example I



$$x_0 = 0(x_0) \oplus 0\downarrow 0 \oplus 1(x_1) \oplus 1\downarrow 0$$
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Solve the system and take the *least* solution:

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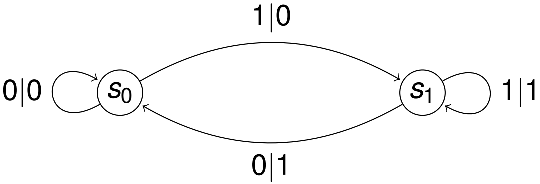

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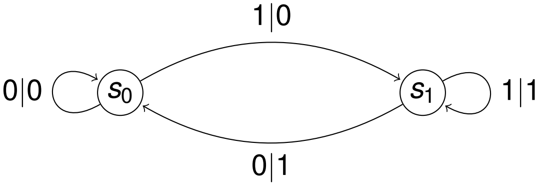

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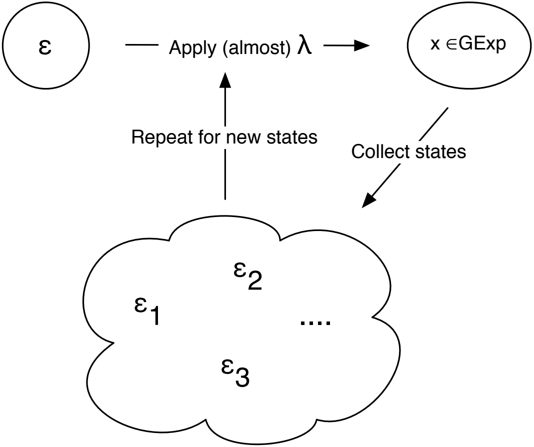

Proof by example II

$$\varepsilon = \mu x.r\langle a(r\langle b(x)\rangle)\rangle \oplus l\langle 1\rangle$$

$$\varepsilon \xrightarrow{\lambda_a} \langle 1, r\langle b(\varepsilon)\rangle\rangle \xrightarrow{\lambda_b} \langle 1, \varepsilon\rangle$$

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\varepsilon \arrow[r, "\lambda_a"] \arrow[dr, "\lambda_b"'] & \langle 1, r\langle b(\varepsilon)\rangle\rangle \arrow[r, "\lambda_b"] \arrow[d, "\lambda_a"] & \langle 1, \varepsilon\rangle \\
 & \langle 0, \emptyset\rangle &
\end{tikzcd}$$

Proof by example II

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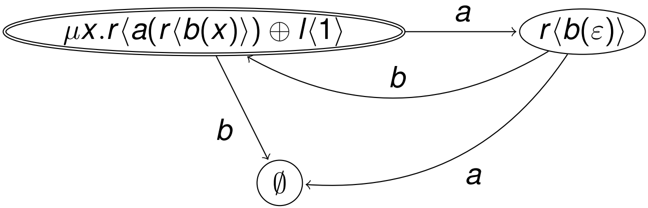

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We need **ACI**!

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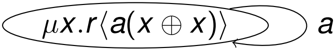

Conclusions and Future work

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- Language of regular expressions for Kripke polynomial coalgebras
- Generalization of Kleene theorem and Kleene algebra

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- Axiomatization of the language
- Automation: `Circ` — Coinductive prover

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Axiomatization

$$\left.\begin{array}{lcl} \varepsilon_1 \oplus \varepsilon_2 & = & \varepsilon_2 \oplus \varepsilon_1 \\ \varepsilon_1 \oplus (\varepsilon_2 \oplus \varepsilon_3) & = & (\varepsilon_1 \oplus \varepsilon_2) \oplus \varepsilon_3 \\ \varepsilon_1 \oplus \varepsilon_1 & = & \varepsilon_1 \\ \varepsilon \oplus \emptyset & = & \varepsilon \end{array}\right\} G$$

$$\left.\begin{array}{lcl} \mu x.\gamma & = & \gamma[\mu x.\gamma/x] \\ \gamma[\varepsilon/x] \leq \varepsilon & \Rightarrow & \mu x.\gamma \leq \varepsilon \end{array}\right\} FP$$

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Sound and complete w.r.t $\sim$

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Similar for $G_1 + G_2$ and G^A

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Axiomatization – example

LTS expressions – $G = 1 + (\mathcal{P} Id)^A$

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No rule

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