Layer by Layer: combining monads

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Motivation: ProbNetKAT

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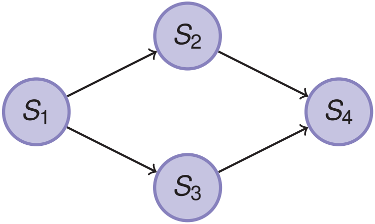

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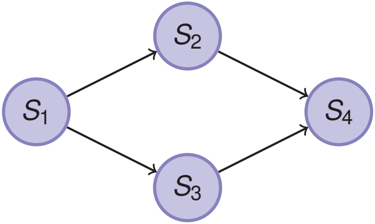

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t =& (sw = S_1; pt = 2; (sw \leftarrow S_2; pt \leftarrow 1) \oplus_{.9} \texttt{drop}) \\
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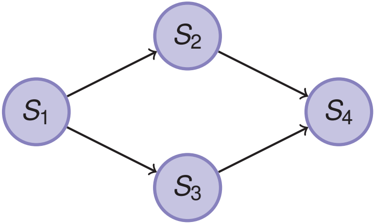

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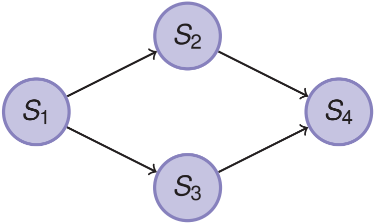

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A packet reaches S_4: $(t; p)^*; (sw = S_4)$

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- General question:

  How can we add features in a principled and controllable manner?

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- No distributive law $\mathcal{PD} \to \mathcal{DP}$
- But there exists a distributive law $(-)^*\mathcal{P} \to \mathcal{P}(-)^*$
- How do we deal with this systematically?

This paper

A general and modular approach for determining :

(a) if a monad combination by distributive law is possible;
(b) if it is not possible, exactly which features are broken by the extension; and
(c) suggests a way to fix the composition by modifying one of the monads.

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- Monads: a categorical way to encode computational effects: Non-determinism, probabilities, side-effects . . .
- Applications of monads include programming language semantics, automata theory, etc.
- It is convenient to compositionally *combine* several effects.

Definitions

Definition

A Monad (T, η, μ) on a category C is:

- An endofunctor $T : C \to C$
- A natural transformation $\eta : 1 \to T$
- A natural transformation $\mu : TT \to T$

(Verifying some structural properties)

We will consider monads on Set.

Examples

Finite Powerset	$\mathcal{P}(A) = \{B \mid B \subseteq A, B \text{ finite}\}$
Free Monoid (List)	$A^* = \{w_1 \dots w_n \mid n \in \mathbb{N}, w_i \in A\}$
Distributions	$\mathcal{D}(A) = \{f \mid f \text{ probability distribution on } A,$ and $Supp(f)$ finite$\}$

Algebras

Definition

An *algebra* for the monad T is an object A together with a morphism $\alpha : TA \to A$.
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Definition

For a signature Σ and a set of equations E we can define a monad T such that $\mathbf{EM}(T) \simeq \mathbf{Alg}(\Sigma, E)$

Examples

	Σ	E

$\mathcal{P}$	`0, +`	`x+0=0+x=x`
		`x+y=y+x`
		`(x+y)+z=x+(y+z)`
		`x+x=x`
		(join-semilattice)

$(-)^*$	`1, ;`	`x;1=1;x=x`
		`(x;y);z=x;(y;z)`
		(monoid)

S, T monads, $\mathbf{EM}(T) \simeq \mathbf{Alg}(\Sigma_T, E_T)$, $\mathbf{EM}(S) \simeq \mathbf{Alg}(\Sigma_S, E_S)$

Definition

A *distributive law* of T over S is a natural transformation $\lambda : ST \to TS$ (verifying structural properties)

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If T distributes over S, then:

- TS is a monad

$$\begin{tikzcd} X \arrow[r, "\eta^T_X"] \arrow[rr, bend right, "u"'] & TX \arrow[r, "\eta^S_{TX}"] & STX \end{tikzcd} \qquad \begin{tikzcd} STSTX \arrow[r, "S\lambda_{TX}"] \arrow[rrr, bend right, "m"'] & SSTTX \arrow[r, "\mu^S_{TTX}"] & STTX \arrow[r, "S\mu^T_X"] & STX \end{tikzcd}$$

- Operations in Σ_S distribute over those of Σ_T

We call S the *inner layer*, T the *outer layer*.

Remarks and questions:

- Distributive laws are one of the go-to methods to **compose monads**
- Implements a one-way distributivity of algebraic operations
- For two given monads, how to know whether there exists a distributive law?
- How to build it?

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Let T be a monoidal monad, then for any finitary signature Σ, there exists a distributive law $\lambda_\Sigma : H_\Sigma T \to TH_\Sigma$ of the polynomial functor associated with Σ over T.

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Theorem

If $\widehat{T}$ sends (Σ, E)-algebras to (Σ, E)-algebras, then it defines a distributive law $\lambda : ST \to TS$.

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- If it commutes then $(TA, \hat{\bullet} : (TA)^2 \to TA) \models x \hat{\bullet} x = x$.
- If it doesn't, we know exactly where the obstacle is and can troubleshoot accordingly.

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- Completely consistent with the semantics of ProbNetKAT

Theorem

Let T be a commutative, relevant and affine monad. For all u and v, T preserves u = v.

Fixing composition – Method 1: changing the inner layer

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$\mathbf{EM}(S) \simeq \mathbf{Alg}(\Sigma_S, E_S)$, $\mathbf{EM}(T) \simeq \mathbf{Alg}(\Sigma_T, E_T)$. Let E'_S be the subset of E_S containing the equations preserved by T.

- Obtain S' from $\mathbf{Alg}(\Sigma_S, E'_S)$
- Compose T with S', obtain a (Σ, E) algebra, where:

$$\Sigma = (\Sigma_T \cup \Sigma_S)$$

$$E = (E_T \cup E'_S \cup \text{ distributivity of } \Sigma_S \text{ over } \Sigma_T)$$

Method 1: fix the inner layer

Example

$\mathcal{D}$ does not preserve idempotency nor distributivity. Drop them and obtain a (Σ, E) algebra where $\Sigma = \{;, 1, +, 0, \oplus_\lambda\}$ and $E =$

- associativity, commutativity, unit laws for $+$
- equations of $(-)^*$
- absorption $\mathrm{p}; 0 = 0; \mathrm{p} = 0$
- equations of $\mathcal{D}$ (convex algebras)
- $p \,;\, (q \oplus_\lambda r) = (p \,;\, q) \oplus_\lambda (p \,;\, r)$
- $(q \oplus_\lambda r) \,;\, p = (q \,;\, p) \oplus_\lambda (r \,;\, p)$
- $p + (q \oplus_\lambda r) = (p + q) \oplus_\lambda (p + r)$
- $(q \oplus_\lambda r) + p = (q + p) \oplus_\lambda (r + p)$

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Two options to fix $\mathcal{P}\mathcal{D}$:

1. Build a monad *PD* that preserves the relevant equations.
2. Replace $\mathcal{P}$ by $\mathcal{P}_c$ and then composition works: $\mathcal{P}_c\mathcal{D}$.

Conclusions

- A principled approach to constructing (equational) languages layer by layer.
- Conditions on existence of distributive laws and potential fixing strategies.
- Note: other troubleshooting strategies are possible!

Thanks

Thank you! Questions?