Programming and Reasoning with Kleene Algebra with Tests

Part 1: Fundamentals

(Nate Foster + Dexter Kozen + Alexandra Silva)*

Plan for today

- Kleene Algebra (with Tests): syntax, semantics, and automata (45min)
- Examples of program equivalence reasoning (30 min)
- Extensions: KAT+B!, NetKAT (90min)

Context

```
while a & b do
      p;
od;
while a do
      q ;
      while a & b do
            p;
      od
od
```

≡ ?

```
while a do
      if b then
            p;
      else
      q;
od
```

Historical remarks

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- Flowchart schemes [Ianov 1960, Manna 1972, …]
- Regular expressions = finite automata [Kleene 1956]
- Completeness of equivalence [Salomaa 1966]

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- Decision procedure PSPACE [Stockmeyer, Meyer 1974]

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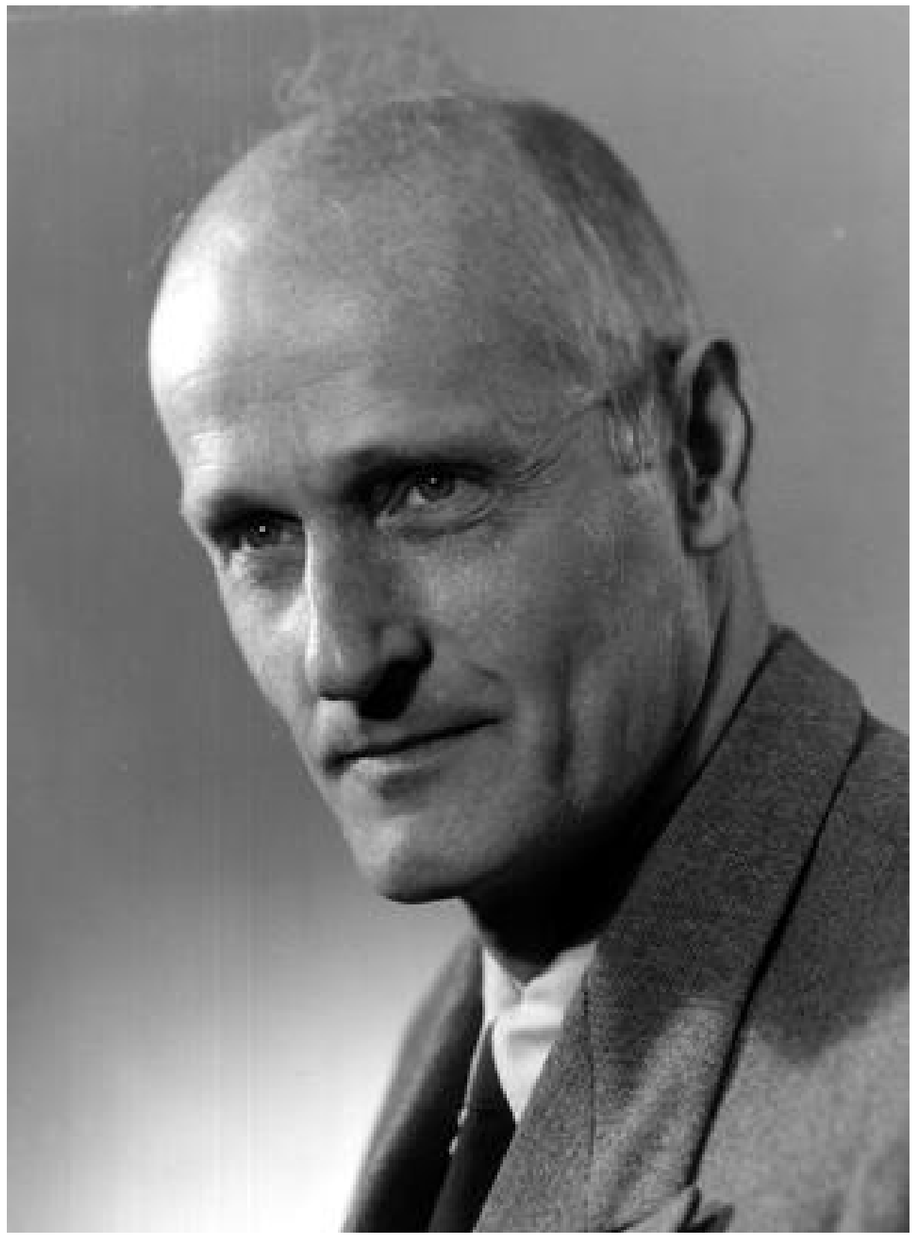
- Completeness of equivalence [Salomaa 1966]
- Algebraic Theory [Conway 1971, Kozen 1990]
- Decision procedure PSPACE [Stockmeyer, Meyer 1974]
- KAT and program schematology [Angus, Kozen 2010]
- NetKAT [Anderson et al 2015]
- GKAT [Smolka et al 2020]

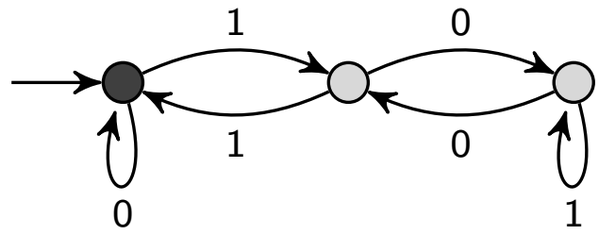
Kleene Algebra



Stephen Cole Kleene
(1909–1994)

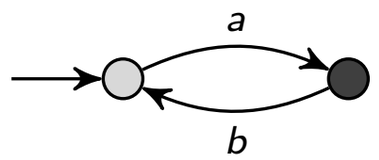
$(0 + 1(01^*0)^*1)^*$

{multiples of 3 in binary}



$(ab)^*a = a(ba)^*$

$\{a, aba, ababa, \ldots\}$



$(a + b)^* = a^*(ba^*)^*$

{all strings over $\{a, b\}$}

Kleene Algebra - syntax

```
e,f :=  0              abort
      | 1              skip
      | a              action
      | e + f          choice
      | e ; f          sequential
      | e*             iteration
```

Kleene Algebra - examples

```
(1+a);(1+b)  – a or b or ab

(a+b)*       – any sequence of a’s and b’s

(a*b)*a*     – ?
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$$(a+b)^* = (a^*b)^*a^*$$

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(a*b)*a*     – ?
```

(a+b)* = (a*b)*a*

Standard Semantics

Regular Sets over A*

sem(0) = { }

sem(1) = {ε}

sem(a) = {a}

sem(e+f) = sem(e) $\cup$ sem(f)

sem(e;f) = sem(e) . sem(f)

sem(e*) = $\bigcup_n$ sem(e)^n

Standard Semantics

Regular Sets over A*

$$sem(0) = \{\ \}$$

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Empty word

$$sem(a) = \{a\}$$

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Standard Semantics

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\mathtt{sem}(e^*) &= \bigcup_n \mathtt{sem}(e)^n
\end{aligned}$$

Empty word

pointwise concatenation
$L.K = \{xy \mid x \in L, y \in K\}$

Equivalence

$$(a+b)^* \equiv (a^*b)^*a^*$$

$$sem((a+b)^*) = sem((a^*b)^*a^*)$$

Equivalence

Syntactic equivalence

$$(a+b)^* \equiv (a^*b)^*a^*$$

$$sem((a+b)^*) = sem((a^*b)^*a^*)$$

Kleene's theorem

Let L be a regular language. TFAE:

1. L = sem(e), for some regular expression e
2. L is accepted by a deterministic finite automaton

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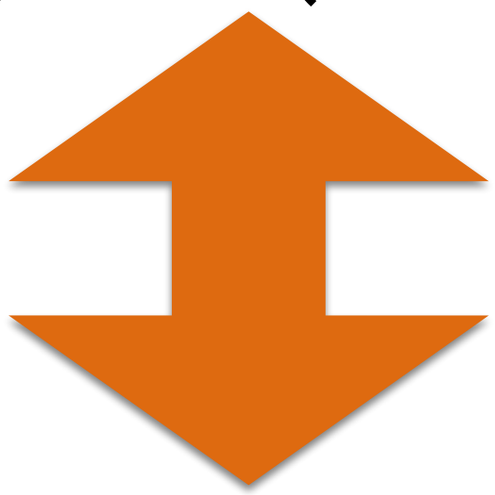

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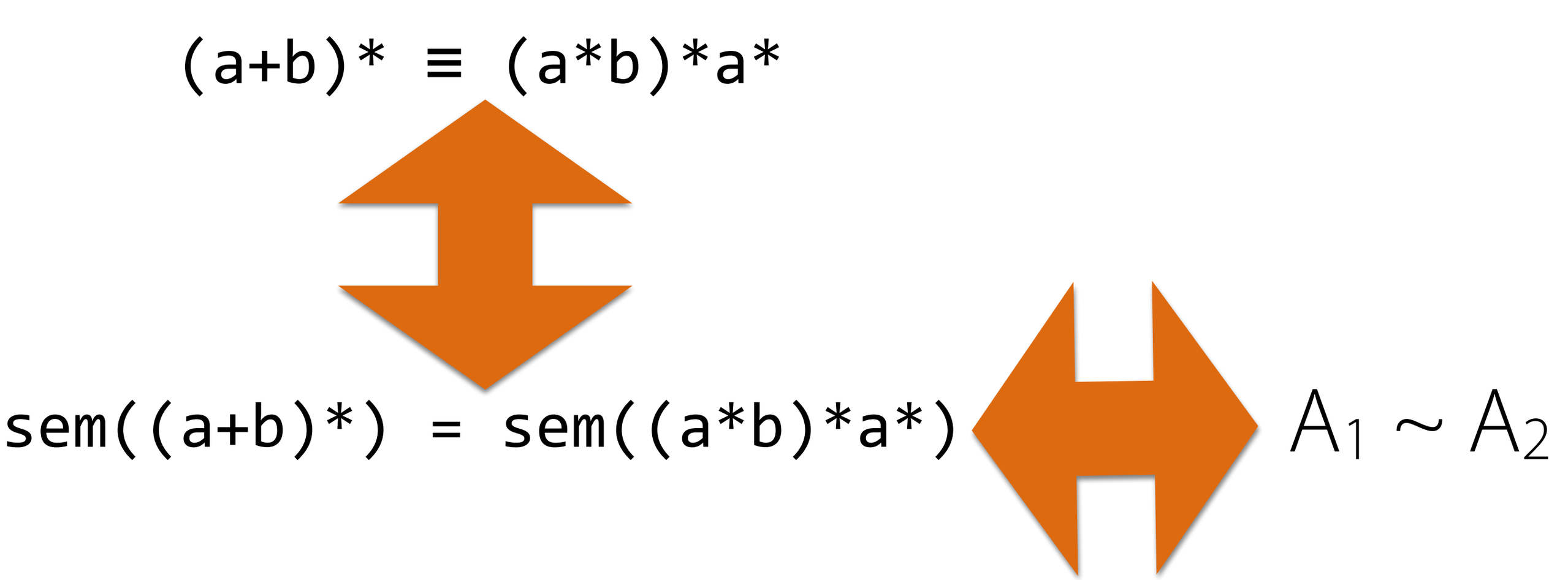

Fundamental Questions

➡ Can we axiomatize the valid equations?

➡ Can we decide if expressions are equivalent?

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Program verification via Equivalence

Axioms

```
Z: e + 0 ≡ e
C: e + f ≡ f + e
A: (e + f) + g ≡ e + (f + g)
I: e + e ≡ e
```

```
SN: e;1 ≡ 1;e ≡ e
SZ: e;0 ≡ 0;e ≡ 0
SA: p;(q;r) ≡ (p;q);r
SDl: p;(q + r) = p;q + p;r
SDr:(p + q);r = p;r + q;r
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idempotent semiring

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A: (e + f) + g ≡ e + (f + g)

I: e + e ≡ e

*** is a least fix point**

FP: 1 + a;a* ≡ a*

1 + a*;a ≡ a*

LFP: f + ex ≤ x

→ e*f ≤ x

SN: e;1 ≡ 1;e ≡ e

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SDl: p;(q + r) = p;q + p;r

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Kleene Algebra, formally

A Kleene algebra is a tuple (**K**, **+**, **;** , ***** , **0**, **1**) satisfying:

Z: e + 0 ≡ e

C: e + f ≡ f + e

A: (e + f) + g ≡ e + (f + g)

I: e + e ≡ e

SN: e;1 ≡ 1;e ≡ e

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Kleene Algebra, examples

- Regular languages
    - **+ is union**
    - **; is pointwise concatenation**
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  - **; is relational composition**
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  - *** is reflexive transitive closure**
- Square Matrices over a KA **K**
  - **+ and ; lifted to usual matrices ops**
  - *** iteratively from 2 x 2**

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$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* = \begin{bmatrix} (a+bd^*c)^* & (a+bd^*c)^*bd^* \\ (d+ca^*b)^*ca^* & (d+ca^*b)^* \end{bmatrix}$$

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Important for relational verification

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* = \begin{bmatrix} (a+bd^*c)^* & (a+bd^*c)^*bd^* \\ (d+ca^*b)^*ca^* & (d+ca^*b)^* \end{bmatrix}$$

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Important for relational verification

Important for automata representations

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* = \begin{bmatrix} (a+bd^*c)^* & (a+bd^*c)^*bd^* \\ (d+ca^*b)^*ca^* & (d+ca^*b)^* \end{bmatrix}$$

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Some results on KA

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➡ Can we devise more efficient procedures?

YES, e.g. antichains [Doyen et al, 2010] or up-to bisimulation [Bonchi-Pous, 2015]

Kleene Algebra - applications

➡ Design and analysis of algorithms
- shortest paths
- connectivity

➡ Program logic and verification
- program analysis
- optimization

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Missing in KA: **Boolean** tests for control structures

Adding tests

(**K**, **B**, **+**, **;**, *, **~**, **0**, **1**)

- (**K**, **+**, **;**, *, **0**, **1**) is a Kleene algebra
- (**B**, **+**, **;** , **~**, **0**, **1**) is a Boolean algebra
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```
if b then p else q = b;p + ~b;q
while b do p = (b;p)*~b
```

Axioms of Boolean Algebra

$$a + 0 \equiv a \qquad a + 1 \equiv 1$$

$$a + b \equiv b + a \qquad a + a \equiv a$$

$$(a + b) + c \equiv a + (b + c)$$

$$\sim(b+c) \equiv \sim b \;;\sim c$$

$$\sim (\sim a) \equiv a$$

$$\sim(a;b) \equiv \sim a + \sim b$$

$$a;1 \equiv a \qquad a;0 \equiv 0 \qquad a;b \equiv b;a$$

$$a;(b;c) \equiv (a;b);c$$

$$a;(b + c) \equiv a;b+a;c \qquad (a + b);c \equiv a;c+b;c$$

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$a + 0 \equiv a$ $\quad$ $a + 1 \equiv 1$ $\quad$ $\sim(b+c) \equiv \sim b\ ;\sim c$

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$(a + b) + c \equiv a + (b + c)$ $\quad$ $\sim(a;b) \equiv \sim a + \sim b$

$a;1 \equiv a$ $\quad$ $a;0 \equiv 0$ $\quad$ $a;b \equiv b;a$

c $\quad$ $(a + b);c \equiv a;c+b;c$

+ and ; have double meaning

Guarded Strings

Σ - action symbols T - set of tests

B - Free Boolean algebra generated by T

At = atoms of B = {α,β,...}

$\alpha_0 p_0 \alpha_1 p_1 \alpha_2 \cdots \alpha_{n-1} p_{n-1} \alpha_n \in (\mathrm{At} \cdot \Sigma)^* \cdot \mathrm{At}$

state of program variables

KAT - semantics

$$\begin{aligned}
sem(b) &= \{\alpha \mid \alpha \leq b\} \\
sem(p) &= \{\alpha p \beta \mid \alpha,\beta \text{ atoms}\} \\
sem(e+f) &= sem(e) \cup sem(f) \\
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\end{aligned}$$

guarded concatenation

$L.K = \{x\alpha y \mid x\alpha \in L, \alpha y \in K\}$

Examples

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if b then p else q = b;p + ~b;q
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```
sem(b;p + ~b;q)
= {αpβ|α <= b} U {αqβ|α <= ~b}
```

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if b then p else q = b;p + ~b;q
while b do p = (b;p)*~b
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sem(b;p + ~b;q)

= {αpβ|α <= b} U {αqβ|α <= ~b}

sem((b;p)*~b)

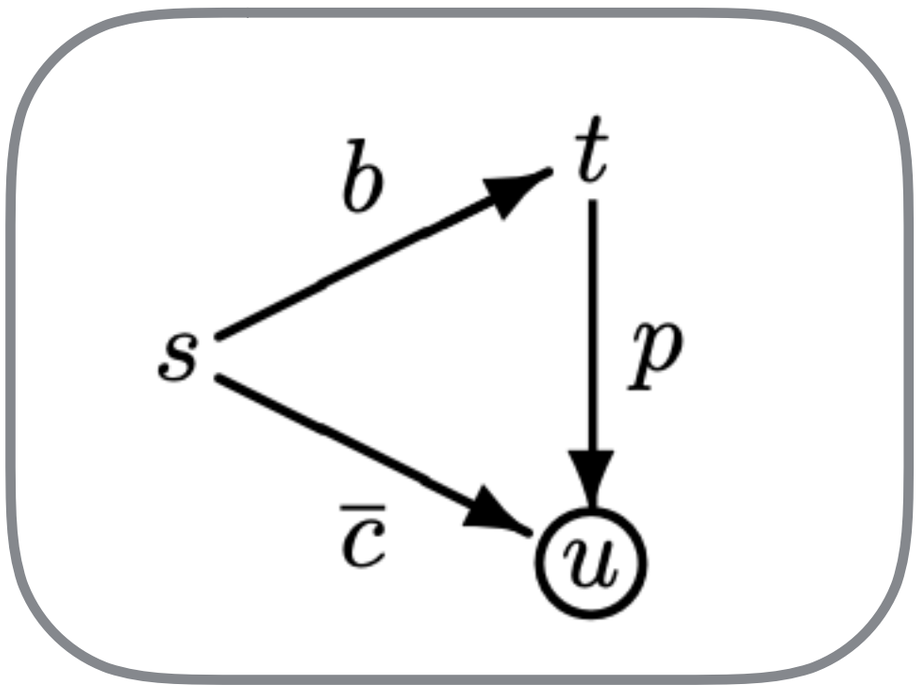
= {α|α <= ~b} U {αpβ | α<=b, β<=~b}

U {$α_0$p$α_1$pβ | $α_i$<=b, β<=~b} U …

Automata

$\alpha_0 p_0 \alpha_1 p_1 \alpha_2 \cdots \alpha_{n-1} p_{n-1} \alpha_n \in (At \cdot \Sigma)^* \cdot At$

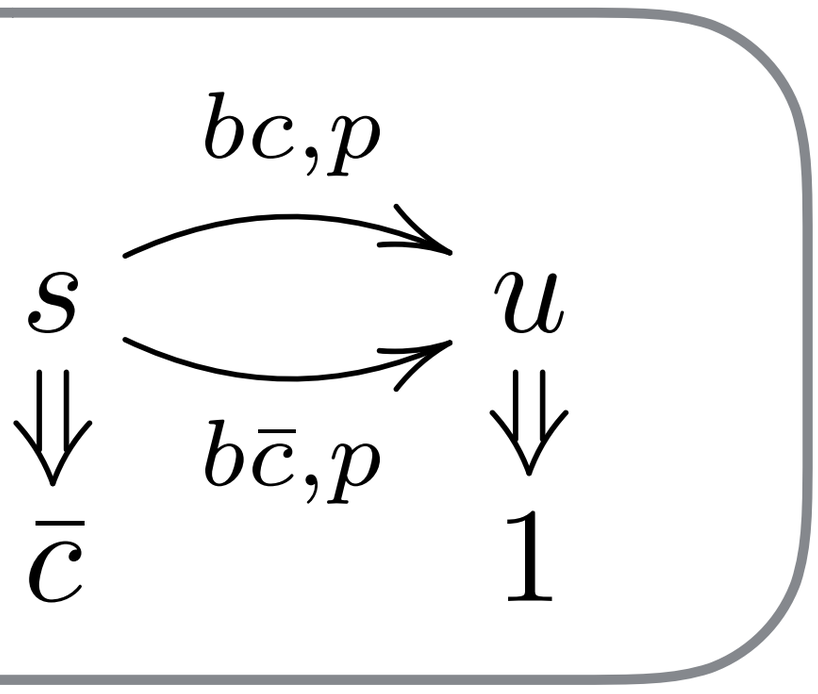
non-deterministic



t: S x (B + Σ) —>P(S)

o: S —> 2

deterministic



t: S x (At x Σ) —>S

o: S —> B

KAT - some results

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➡ Connections to propositional Hoare Logic and PDL

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➡ Connections to propositional Hoare Logic and PDL

```
{b}p{c} <=> bp~c ≡ 0
```

Break exercises

1. x*x* = x*
2. x*=x**
3. (x+y)* = (x*y)*x* **denesting**
4. x(yx)* = (xy)*x **sliding**
5. xy=yz => x*y = yz*

Denesting

$$(x+y)^* = (x^*y)^*x^*$$

Denesting

$$(x+y)^* = (x^*y)^*x^*$$

$1 \leq (x^*y)^*x^*$

$xx^*(yx^*)^* \leq x^*(yx^*)^*$

$yx^*(yx^*)^* \leq (yx^*)^* \leq x^*(yx^*)^*$

Denesting

$$(x+y)^* = (x^*y)^*x^*$$

$1 \le (x^*y)^*x^*$

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monoticity + distributivity

$1 + (x+y)\,x^*(yx^*)^* \le x^*(yx^*)^*$

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monoticity + distributivity

$1 + (x+y)\,x^*(yx^*)^* \le x^*(yx^*)^*$

fixpoint rule

$f + ex \le x \to e^*f \le x$

$$(x+y)^* \le (x^*y)^*x^*$$

KAT - applications

Compiler Optimizations [Kozen & Patron 00]

- dead code elimination
- common subexpression elimination
- copy propagation
- loop hoisting
- **loop unrolling**
- **reducing nesting depth**
- induction variable elimination
- instruction scheduling
- elimination of redundant instructions
- array bounds check elimination

Example - loop denesting

```
while b do {                    if b then {
  p;                      ≡       p;
  while c do q;                   while b ∨ c do {
}                                   if c then q else p
                                  }
                                }
```

(bp(cq)*~c)*~b ≡ bp((b+c)(cq+~cp))*~(b+c) + ~b

Proof

$$(bp(cq)*\sim c)*\sim b \equiv bp((b+c)(cq+\sim cp))*\sim(b+c) + \sim b$$

Proof

$$(bp(cq)^*{\sim}c)^*{\sim}b \equiv bp((b+c)(cq+{\sim}cp))^*{\sim}(b+c) + {\sim}b$$

$$bp((b+c)(cq+{\sim}cp))^*{\sim}(b+c) + {\sim}b$$

Proof

(bp(cq)*~c)*~b ≡ bp((b+c)(cq+~cp))*~(b+c) + ~b

bp((b+c)(cq+~cp))*~(b+c) + ~b

≡ {Distr+Boolean Alg}

Proof

(bp(cq)*~c)*~b ≡ bp((b+c)(cq+~cp))*~(b+c) + ~b

bp((b+c)(cq+~cp))*~(b+c) + ~b

≡ {Distr+Boolean Alg}

bp(bcq+b~cp+cq + c~cq)*~b~c + ~b

Proof

(bp(cq)*~c)*~b ≡ bp((b+c)(cq+~cp))*~(b+c) + ~b

bp((b+c)(cq+~cp))*~(b+c) + ~b

≡ {Distr+Boolean Alg}

bp(bcq+b~cp+cq + c~cq)*~b~c + ~b

≡ {Comm + Dist + Boolean Alg}

Proof

(bp(cq)*~c)*~b ≡ bp((b+c)(cq+~cp))*~(b+c) + ~b

bp((b+c)(cq+~cp))*~(b+c) + ~b

≡ {Distr+Boolean Alg}

bp(bcq+b~cp+cq + c~cq)*~b~c + ~b

≡ {Comm + Dist + Boolean Alg}

bp(cq+~cbp)*~b~c + ~b

Proof

(bp(cq)*~c)*~b ≡ bp((b+c)(cq+~cp))*~(b+c) + ~b

bp((b+c)(cq+~cp))*~(b+c) + ~b

≡ {Distr+Boolean Alg}

bp(bcq+b~cp+cq + c~cq)*~b~c + ~b

≡ {Comm + Dist + Boolean Alg}

bp(cq+~cbp)*~b~c + ~b

≡ {Denesting: (e+f)*=(e*f)*e*}

Proof

(bp(cq)*~c)*~b ≡ bp((b+c)(cq+~cp))*~(b+c) + ~b

bp((b+c)(cq+~cp))*~(b+c) + ~b

≡ {Distr+Boolean Alg}

bp(bcq+b~cp+cq + c~cq)*~b~c + ~b

≡ {Comm + Dist + Boolean Alg}

bp(cq+~cbp)*~b~c + ~b

≡ {Denesting: (e+f)*=(e*f)*e*}

bp((cq)*~cbp)*(cq)*)~b~c + ~b

Proof

$$(bp(cq)^*\sim c)^*\sim b \equiv bp((b+c)(cq+\sim cp))^*\sim(b+c) + \sim b$$

$bp((b+c)(cq+\sim cp))^*\sim(b+c) + \sim b$

$\equiv$ {Distr+Boolean Alg}

$bp(bcq+b\sim cp+cq + c\sim cq)^*\sim b\sim c + \sim b$

$\equiv$ {Comm + Dist + Boolean Alg}

$bp(cq+\sim cbp)^*\sim b\sim c + \sim b$

$\equiv$ {Denesting: $(e+f)^*=(e^*f)^*e^*$}

$bp((cq)^*\sim cbp)^*(cq)^*)\sim b\sim c + \sim b$

$\equiv$ {Sliding: $(ef)^*e=e(fe)^*$}

Proof

$$(bp(cq)^*\sim c)^*\sim b \equiv bp((b+c)(cq+\sim cp))^*\sim(b+c) + \sim b$$

$$bp((b+c)(cq+\sim cp))^*\sim(b+c) + \sim b$$

$$\equiv \{\text{Distr+Boolean Alg}\}$$

$$bp(bcq+b\sim cp+cq + c\sim cq)^*\sim b\sim c + \sim b$$

$$\equiv \{\text{Comm + Dist + Boolean Alg}\}$$

$$bp(cq+\sim cbp)^*\sim b\sim c + \sim b$$

$$\equiv \{\text{Denesting: } (e+f)^*=(e^*f)^*e^*\}$$

$$bp((cq)^*\sim cbp)^*(cq)^*)\sim b\sim c + \sim b$$

$$\equiv \{\text{Sliding: } (ef)^*e=e(fe)^*\}$$

$$(bp(cq)^*\sim c)^*bp(cq)^*\sim b\sim c + \sim b$$

Proof

$$(bp(cq)^*\sim c)^*\sim b \equiv bp((b+c)(cq+\sim cp))^*\sim(b+c) + \sim b$$

$$bp((b+c)(cq+\sim cp))^*\sim(b+c) + \sim b$$

$$\equiv \{\text{Distr+Boolean Alg}\}$$

$$bp(bcq+b\sim cp+cq + c\sim cq)^*\sim b\sim c + \sim b$$

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$$\equiv \{\text{Denesting: } (e+f)^*=(e^*f)^*e^*\}$$

$$bp((cq)^*\sim cbp)^*(cq)^*)\sim b\sim c + \sim b$$

$$\equiv \{\text{Sliding: } (ef)^*e=e(fe)^*\}$$

$$(bp(cq)^*\sim c)^*bp(cq)^*\sim b\sim c + \sim b$$

$$\equiv \{\text{Distributivity}\}$$

Proof

$$(bp(cq)^*\sim c)^*\sim b \equiv bp((b+c)(cq+\sim cp))^*\sim(b+c) + \sim b$$

$$bp((b+c)(cq+\sim cp))^*\sim(b+c) + \sim b$$

$$\equiv \{\text{Distr+Boolean Alg}\}$$

$$bp(bcq+b\sim cp+cq + c\sim cq)^*\sim b\sim c + \sim b$$

$$\equiv \{\text{Comm + Dist + Boolean Alg}\}$$

$$bp(cq+\sim cbp)^*\sim b\sim c + \sim b$$

$$\equiv \{\text{Denesting: } (e+f)^*=(e^*f)^*e^*\}$$

$$bp((cq)^*\sim cbp)^*(cq)^*)\sim b\sim c + \sim b$$

$$\equiv \{\text{Sliding: } (ef)^*e=e(fe)^*\}$$

$$(bp(cq)^*\sim c)^*bp(cq)^*\sim b\sim c + \sim b$$

$$\equiv \{\text{Distributivity}\}$$

$$((bp(cq)^*\sim c)^*bp(cq)^*\sim c + 1)\sim b$$

Proof

$$(bp(cq)^*\sim c)^*\sim b \equiv bp((b+c)(cq+\sim cp))^*\sim(b+c) + \sim b$$

$$bp((b+c)(cq+\sim cp))^*\sim(b+c) + \sim b$$

$$\equiv \{\text{Distr+Boolean Alg}\}$$

$$bp(bcq+b\sim cp+cq + c\sim cq)^*\sim b\sim c + \sim b$$

$$\equiv \{\text{Comm + Dist + Boolean Alg}\}$$

$$bp(cq+\sim cbp)^*\sim b\sim c + \sim b$$

$$\equiv \{\text{Denesting: } (e+f)^*=(e^*f)^*e^*\}$$

$$bp((cq)^*\sim cbp)^*(cq)^*)\sim b\sim c + \sim b$$

$$\equiv \{\text{Sliding: } (ef)^*e=e(fe)^*\}$$

$$(bp(cq)^*\sim c)^*bp(cq)^*\sim b\sim c + \sim b$$

$$\equiv \{\text{Distributivity}\}$$

$$((bp(cq)^*\sim c)^*bp(cq)^*\sim c + 1)\sim b$$

$$\equiv \{\text{FP}\}$$

Proof

$$(bp(cq)^*\sim c)^*\sim b \equiv bp((b+c)(cq+\sim cp))^*\sim(b+c) + \sim b$$

$$bp((b+c)(cq+\sim cp))^*\sim(b+c) + \sim b$$

$$\equiv \{\text{Distr+Boolean Alg}\}$$

$$bp(bcq+b\sim cp+cq + c\sim cq)^*\sim b\sim c + \sim b$$

$$\equiv \{\text{Comm + Dist + Boolean Alg}\}$$

$$bp(cq+\sim cbp)^*\sim b\sim c + \sim b$$

$$\equiv \{\text{Denesting: } (e+f)^*=(e^*f)^*e^*\}$$

$$bp((cq)^*\sim cbp)^*(cq)^*)\sim b\sim c + \sim b$$

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$$(bp(cq)^*\sim c)^*bp(cq)^*\sim b\sim c + \sim b$$

$$\equiv \{\text{Distributivity}\}$$

$$((bp(cq)^*\sim c)^*bp(cq)^*\sim c + 1)\sim b$$

$$\equiv \{\text{FP}\}$$

$$(bp(cq)^*\sim c)^*\sim b$$

Proof

$$(bp(cq)^*\sim c)^*\sim b \equiv bp((b+c)(cq+\sim cp))^*\sim(b+c) + \sim b$$

$$bp((b+c)(cq+\sim cp))^*\sim(b+c) + \sim b$$

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$$bp(bcq+b\sim cp+cq + c\sim cq)^*\sim b\sim c + \sim b$$

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$$bp((cq)^*\sim cbp)^*(cq)^*)\sim b\sim c + \sim b$$

$$\equiv \{\text{Sliding: } (ef)^*e=e(fe)^*\}$$

$$(bp(cq)^*\sim c)^*bp(cq)^*\sim b\sim c + \sim b$$

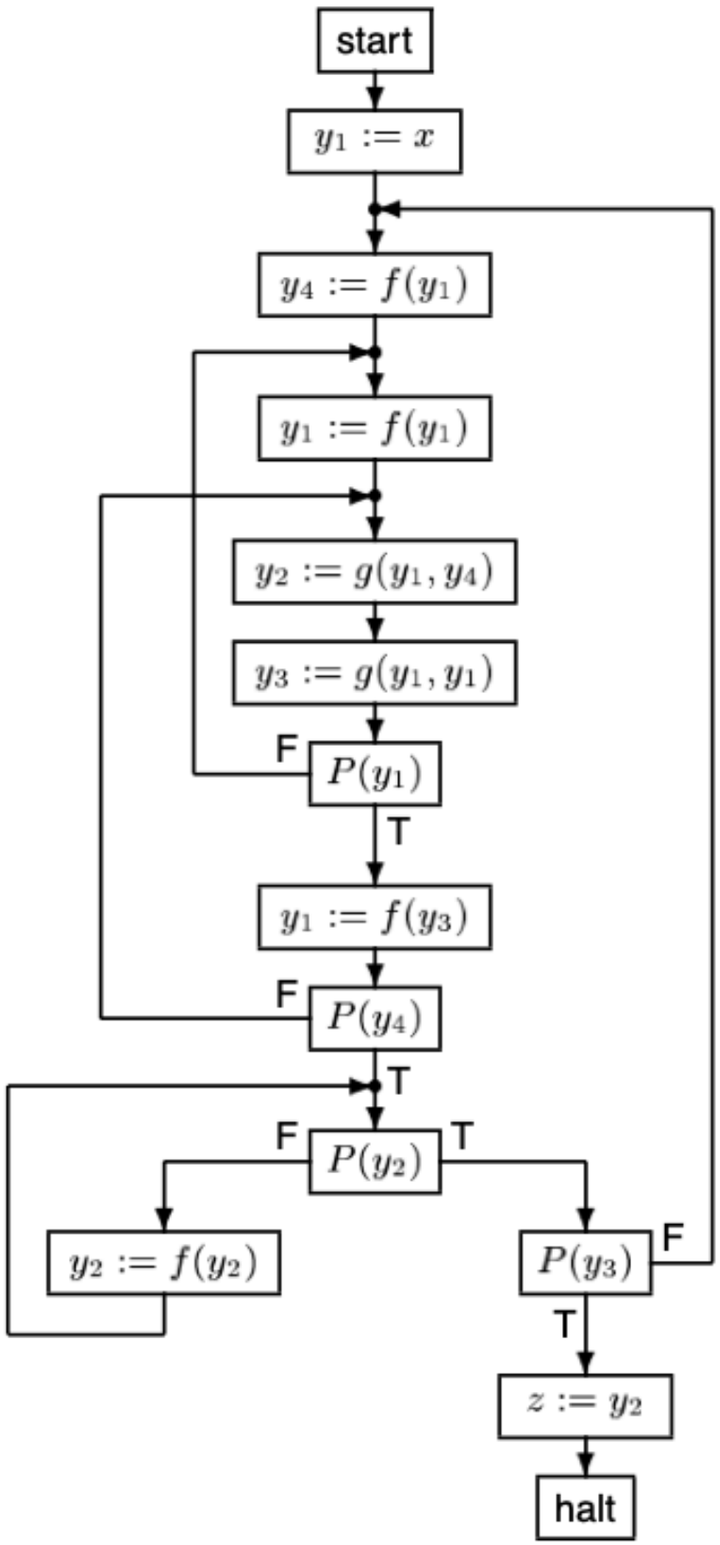
$$\equiv \{\text{Distributivity}\}$$

$$((bp(cq)^*\sim c)^*bp(cq)^*\sim c + 1)\sim b$$

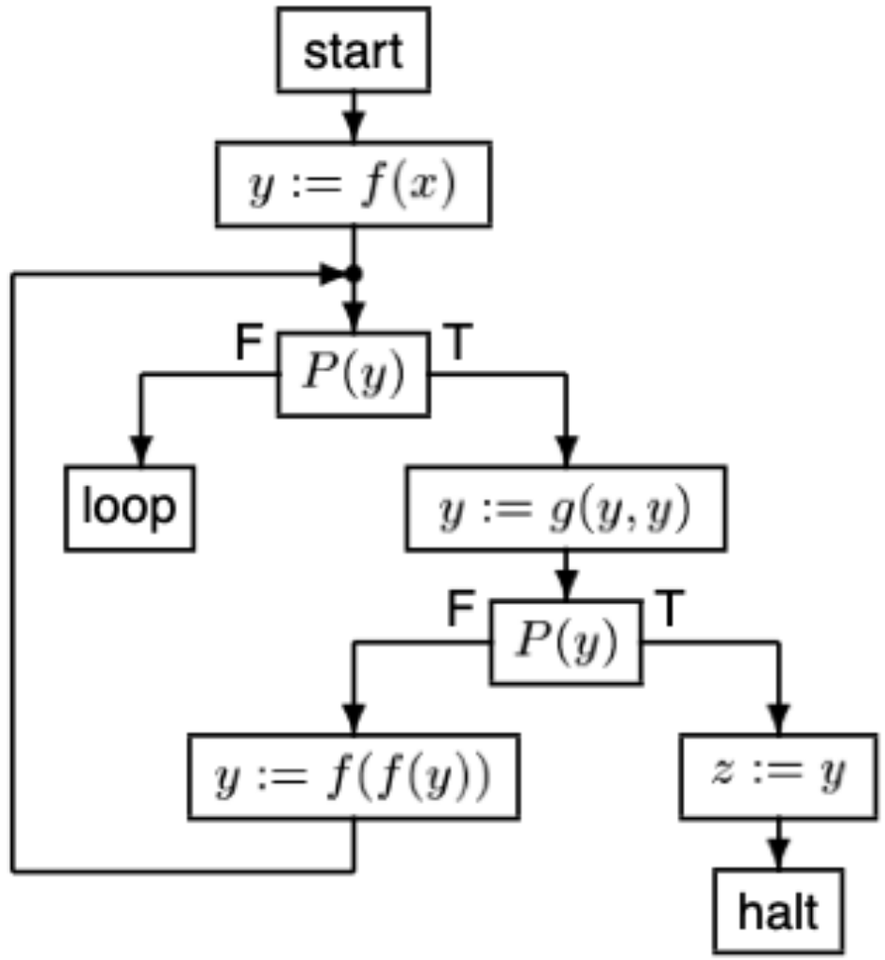
$$\equiv \{\text{FP}\}$$

$$(bp(cq)^*\sim c)^*\sim b$$

Scheme equivalence

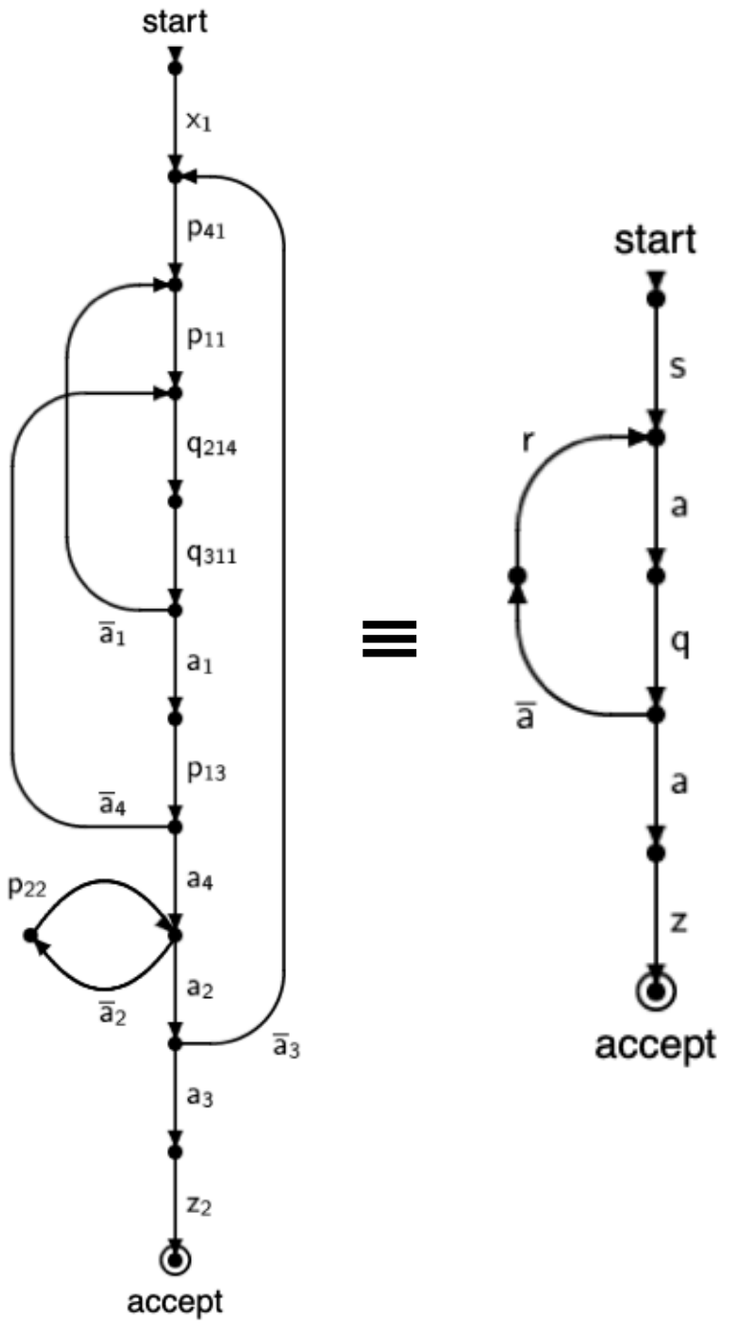


≡



Example of Paterson from [Manna 74]

Scheme equivalence



≡

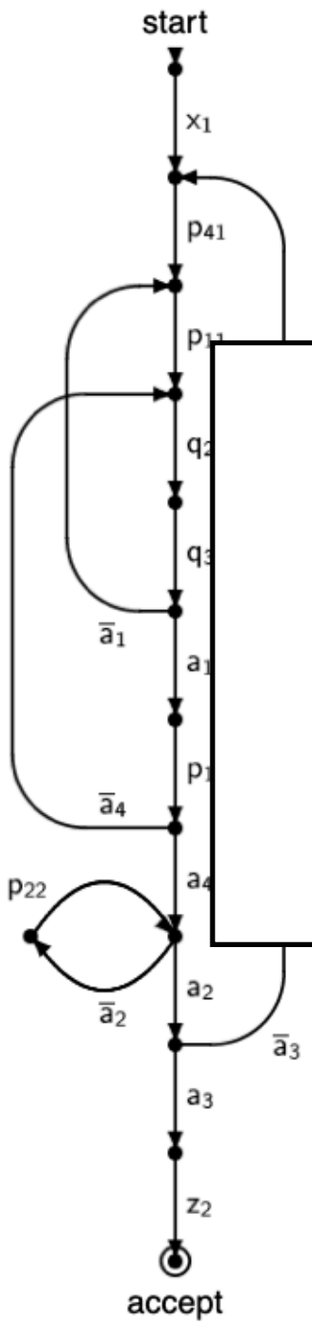
Kleene Theorem

$x_1 p_{41} p_{11} q_{214} q_{311}$
$(\sim a_1 p_{11} q_{214} q_{311})^* a_1 p_{13}$ …
$a_4 (\sim a_2 p_{22})^* a_2 a_3 z_2$

≡

$saq(\sim araq)^* az$

Scheme equivalence



start

$$
\begin{aligned}
p_{41}p_{11}q_{214} &= y_4 := f(y_1)\,;\, y_1 := f(y_1)\,;\, y_2 := g(y_1, y_4) \\
&= y_1 := f(y_1)\,;\, y_4 := y_1\,;\, y_2 := g(y_1, y_4) \\
&= y_1 := f(y_1)\,;\, y_4 := y_1\,;\, y_2 := g(y_1, y_1) \\
&= y_4 := f(y_1)\,;\, y_1 := f(y_1)\,;\, y_2 := g(y_1, y_1) \\
&= p_{41}p_{11}q_{211}.
\end{aligned}
$$

accept

p13 …

Bohm-Jacopini Theorem

Folk Theorem

Every **while** program can be simulated by a **while** program with at most one **while** loop as long as extra Boolean variables are allowed.

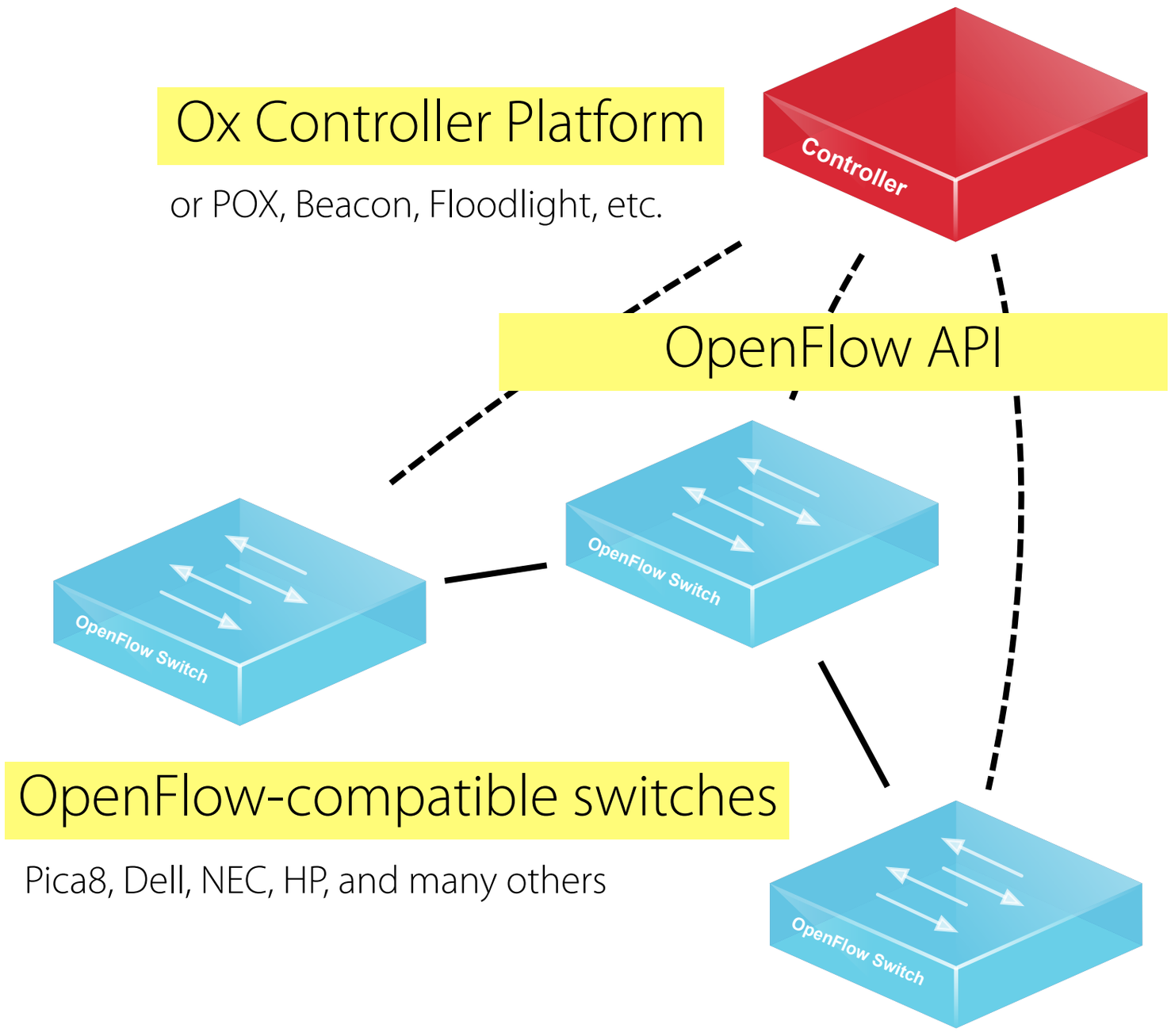
Dexter Kozen. ***Kleene algebra with tests****. ACM Trans. Programming Languages and Systems (TOPLAS), 19(3):427-443, May 1997*

Programming and Reasoning with Kleene Algebra with Tests

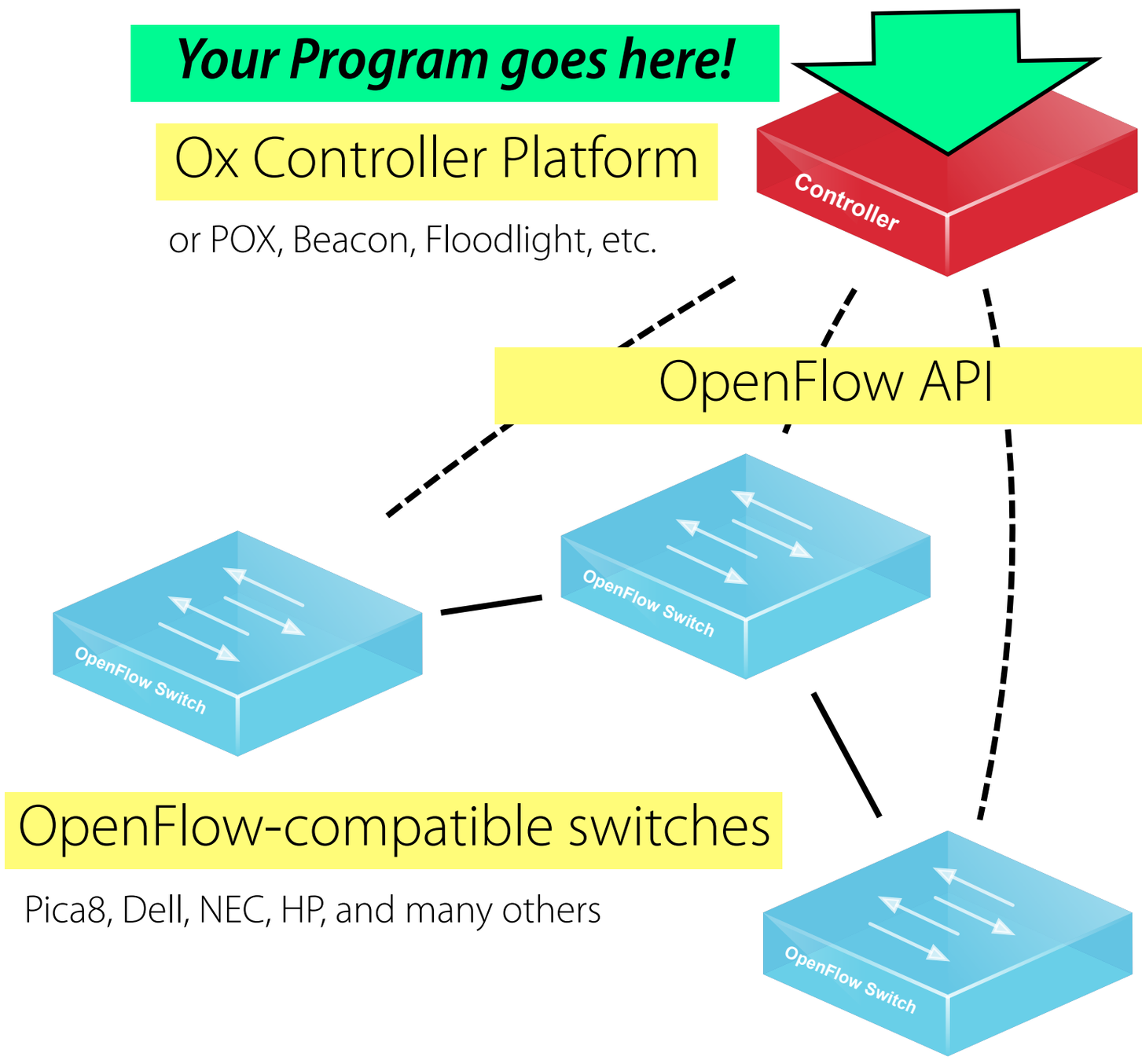
Part 2: Extensions & Applications

(Nate Foster + Dexter Kozen + Alexandra Silva)*

SDN Architecture

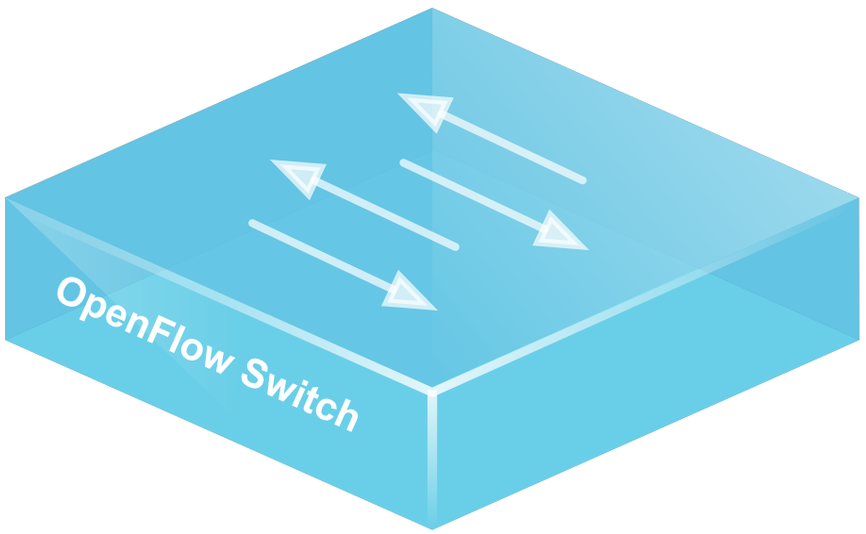


SDN Architecture



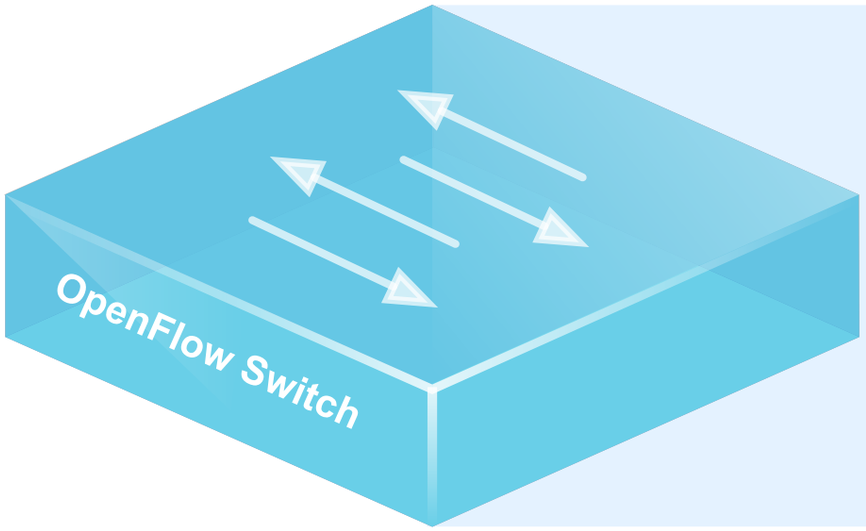
OpenFlow Switch

General-purpose packet-processing device that can be used to implement switches, routers, firewalls, etc.



OpenFlow Switch

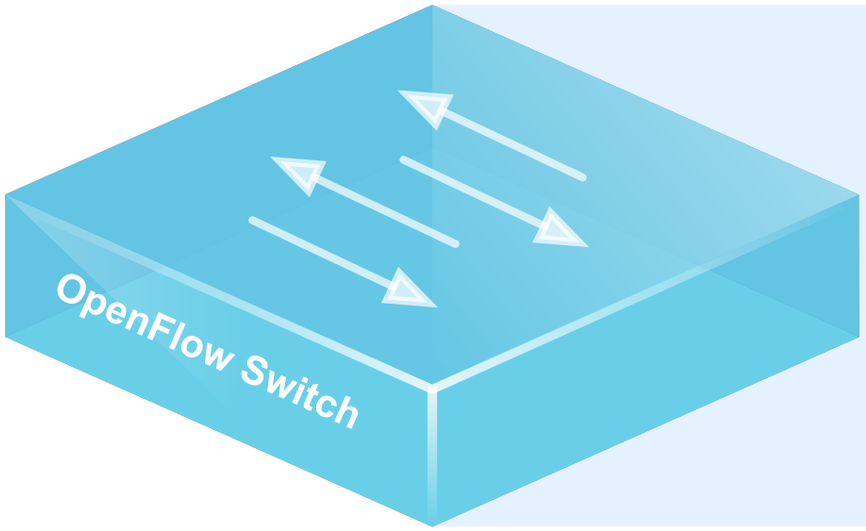
General-purpose packet-processing device that can be used to implement switches, routers, firewalls, etc.



Match	Actions	Counters
10.0.0.1	Drop	(73,2458)
10.0.0.2	Forward 2	(16,846)
10.0.0.3	Forward 3	(23,5729)
*	Controller	(5,472)

OpenFlow Switch

General-purpose packet-processing device that can be used to implement switches, routers, firewalls, etc.



Match	Actions	Counters
10.0.0.1	Drop	(73,2458)
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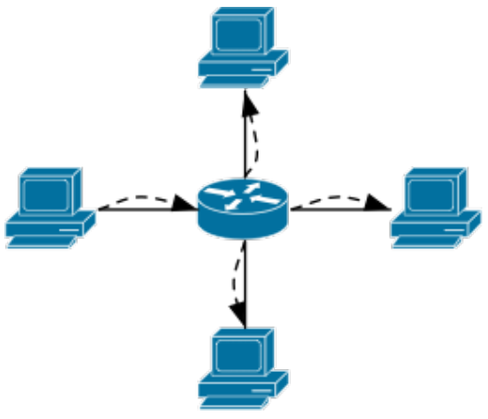
10.0.0.3	Forward 3	(23,5729)
*	Controller	(5,472)

Key data structure is a *flow table* containing a prioritized list of match-action *rules* and *counters*

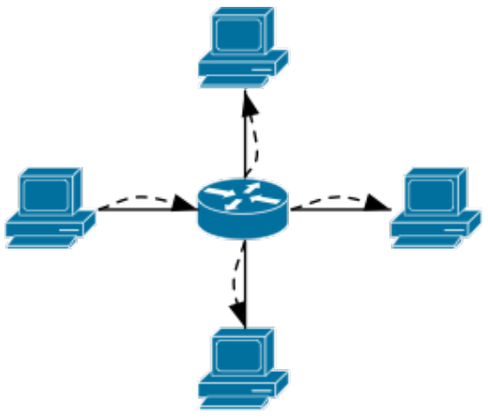
Example

Topology



Example

Topology



Specification

- Forward packets to hosts 1-4
- Monitor traffic to unknown hosts
- Flood broadcast traffic to all hosts
- Disallow SSH traffic from hosts 1-2

Flow Table

```
{pattern={ethSrc=00:00:00:00:00:01,ethTyp=0x800,ipProto=0x06, tcpDstPort=22},action=[]}
{pattern={ethSrc=00:00:00:00:00:02,ethTyp=0x800,ipProto=0x06, tcpDstPort=22},action=[]}
{pattern={ethDst=00:00:00:00:00:01},action=[Output(1)]}
{pattern={ethDst=00:00:00:00:00:02},action=[Output(2)]}
{pattern={ethDst=00:00:00:00:00:03},action=[Output(3)]}
{pattern={ethDst=00:00:00:00:00:04},action=[Output(4)]}
{pattern={ethDst=ff:ff:ff:ff:ff:ff,port=1},action=[Output(4), Output(3), Output(2)]}
{pattern={ethDst=ff:ff:ff:ff:ff:ff,port=2},action=[Output(4), Output(3), Output(1)]}
{pattern={ethDst=ff:ff:ff:ff:ff:ff,port=3},action=[Output(4), Output(2), Output(1)]}
{pattern={ethDst=ff:ff:ff:ff:ff:ff,port=4},action=[Output(3), Output(2), Output(1)]}
{pattern={ethDst=ff:ff:ff:ff:ff:ff},action=[]}
{pattern={},action=[Controller]}
```

Example: Forward

```
let forward =
  if ethDst = 00:00:00:00:00:01 then
    port := 1
  else if ethDst = 00:00:00:00:00:02 then
    port := 2
  else if ethDst = 00:00:00:00:00:03 then
    port := 3
  else if ethDst = 00:00:00:00:00:04 then
    port := 4
  else
    false
```

Example: Broadcast

```
let flood =
  if port = 1 then
    port := 2 + port := 3 + port := 4
  else if port = 2 then
    port := 1 + port := 3 + port := 4
  else if port = 3 then
    port := 1 + port := 2 + port := 4
  else if port = 4 then
    port := 1 + port := 2 + port := 3
  else
    false

let broadcast =
  if ethDst = ff:ff:ff:ff:ff:ff then
    flood
  else
    false
```

Example: Routing

```
let route = forward + broadcast
```

Example: Monitor

```
let monitor =
  if !(ethDst = 00:00:00:00:00:01 +
       ethDst = 00:00:00:00:00:02 +
       ethDst = 00:00:00:00:00:03 +
       ethDst = 00:00:00:00:00:04 +
       ethDst = ff:ff:ff:ff:ff:ff) then
    port := unknown
  else
    false
```

Example: Firewall

```
let firewall =
  if (ethSrc = 00:00:00:00:00:01 +
      ethSrc = 00:00:00:00:00:02) ;
     ethTyp = 0x800 ;
     ipProto = 0x06 ;
     tcpDstPort = 22 then
     false
  else
     true
```

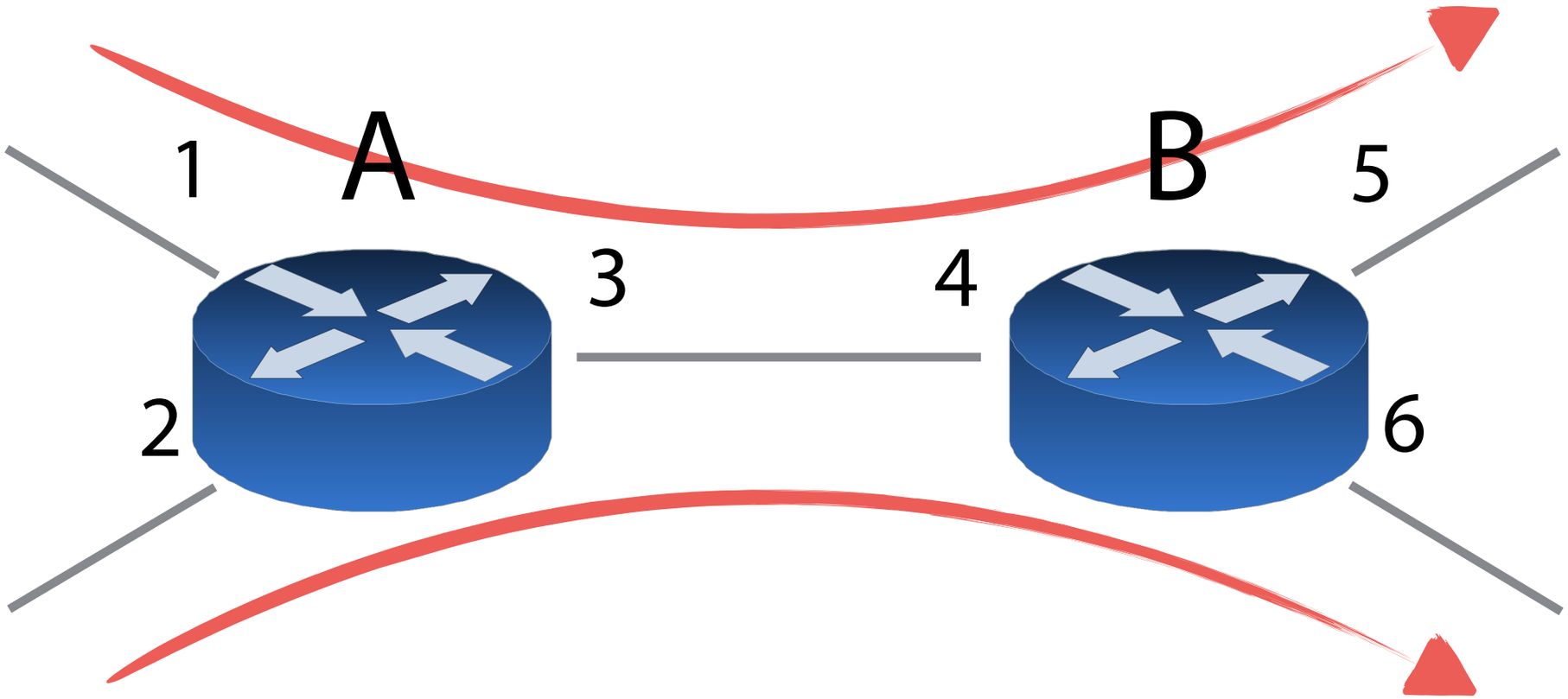
Example: Main Policy

```
let main = (route + monitor); firewall
```

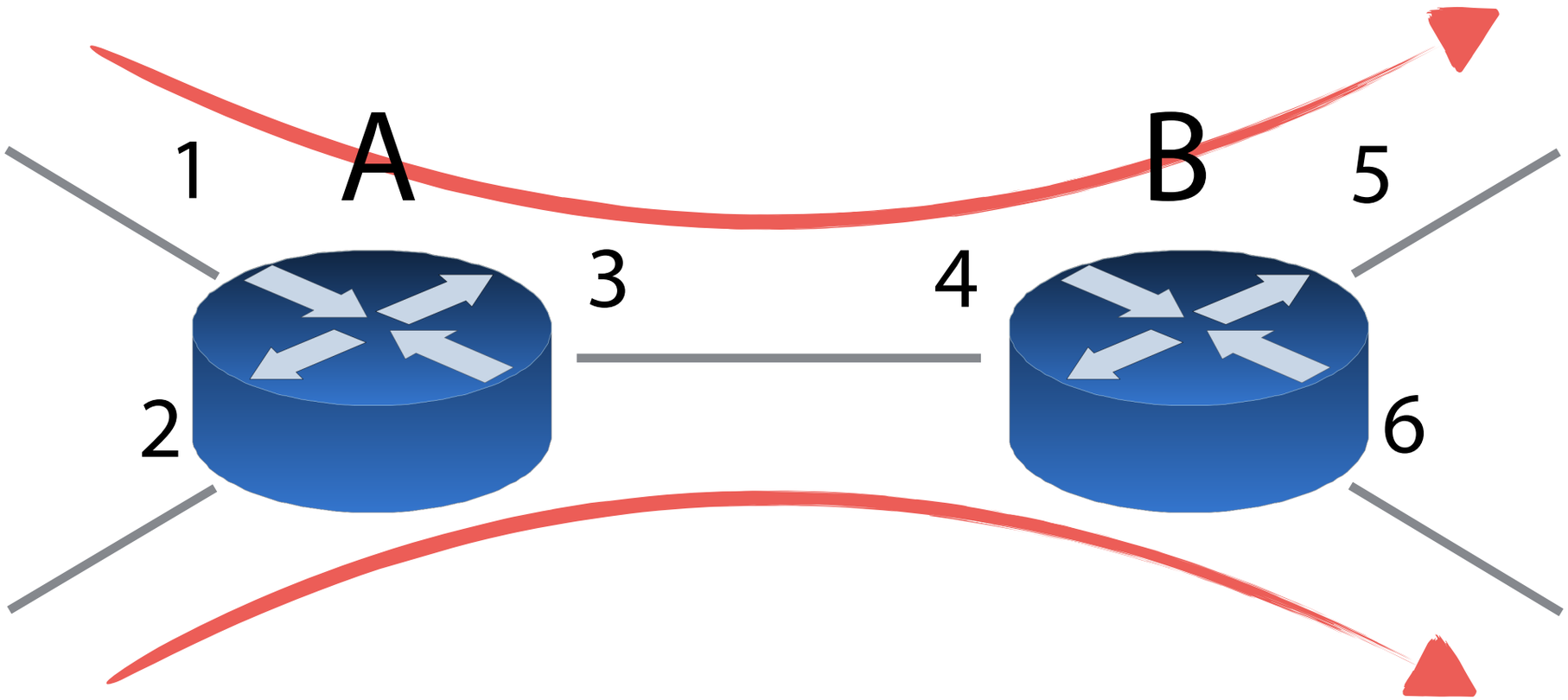
compiles to…

```
{pattern={ethSrc=00:00:00:00:00:01,ethTyp=0x800,ipProto=0x06, tcpDstPort=22},action=[]}
{pattern={ethSrc=00:00:00:00:00:02,ethTyp=0x800,ipProto=0x06, tcpDstPort=22},action=[]}
{pattern={ethDst=00:00:00:00:00:01},action=[Output(1)]}
{pattern={ethDst=00:00:00:00:00:02},action=[Output(2)]}
{pattern={ethDst=00:00:00:00:00:03},action=[Output(3)]}
{pattern={ethDst=00:00:00:00:00:04},action=[Output(4)]}
{pattern={ethDst=ff:ff:ff:ff:ff:ff,port=1},action=[Output(4), Output(3), Output(2)]}
{pattern={ethDst=ff:ff:ff:ff:ff:ff,port=2},action=[Output(4), Output(3), Output(1)]}
{pattern={ethDst=ff:ff:ff:ff:ff:ff,port=3},action=[Output(4), Output(2), Output(1)]}
{pattern={ethDst=ff:ff:ff:ff:ff:ff,port=4},action=[Output(3), Output(2), Output(1)]}
{pattern={ethDst=ff:ff:ff:ff:ff:ff},action=[]}
{pattern={},action=[Controller]}
```

Example: Paths



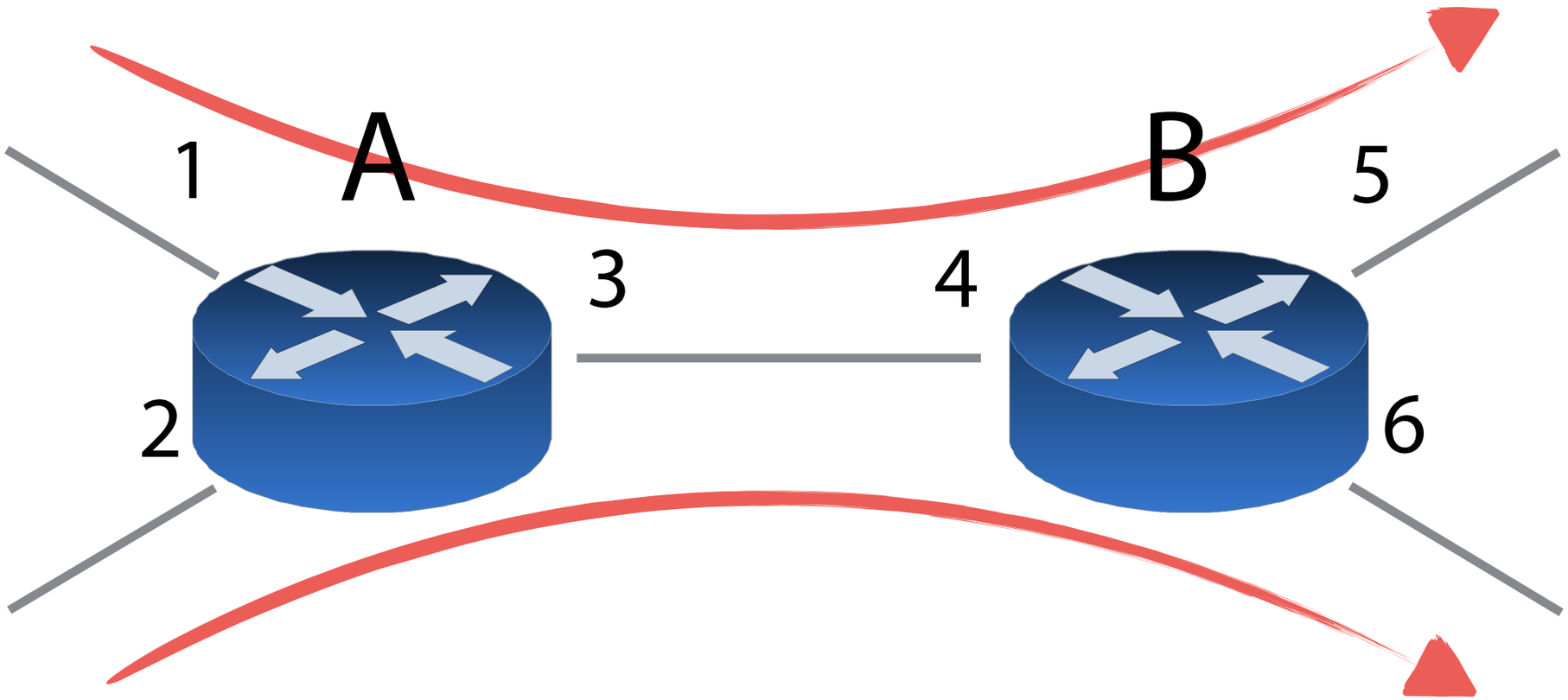
Local Program



pol_A

pol_B

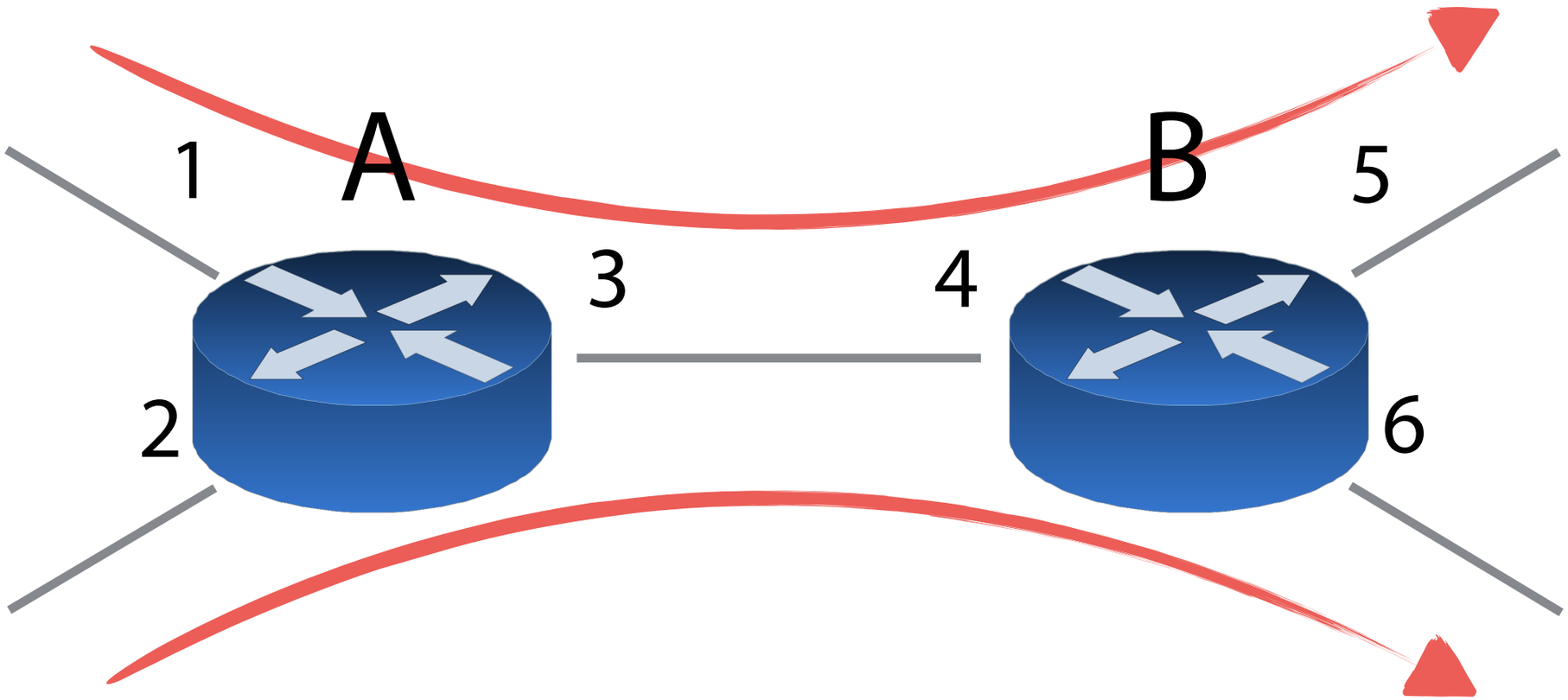
Local Program



port:=3

???

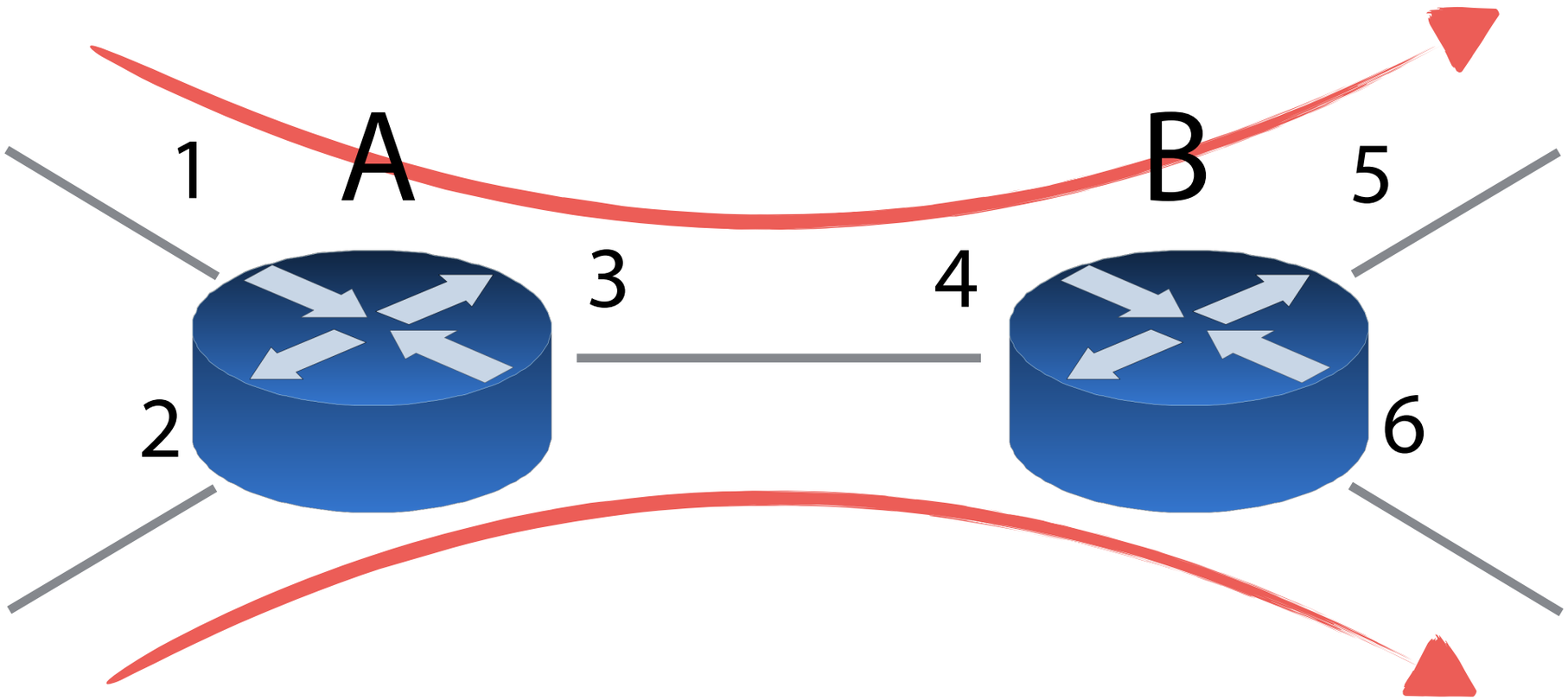
Local Program



port=1; tag:=1; port:=3
+
port=2; tag:=2; port:=3

???

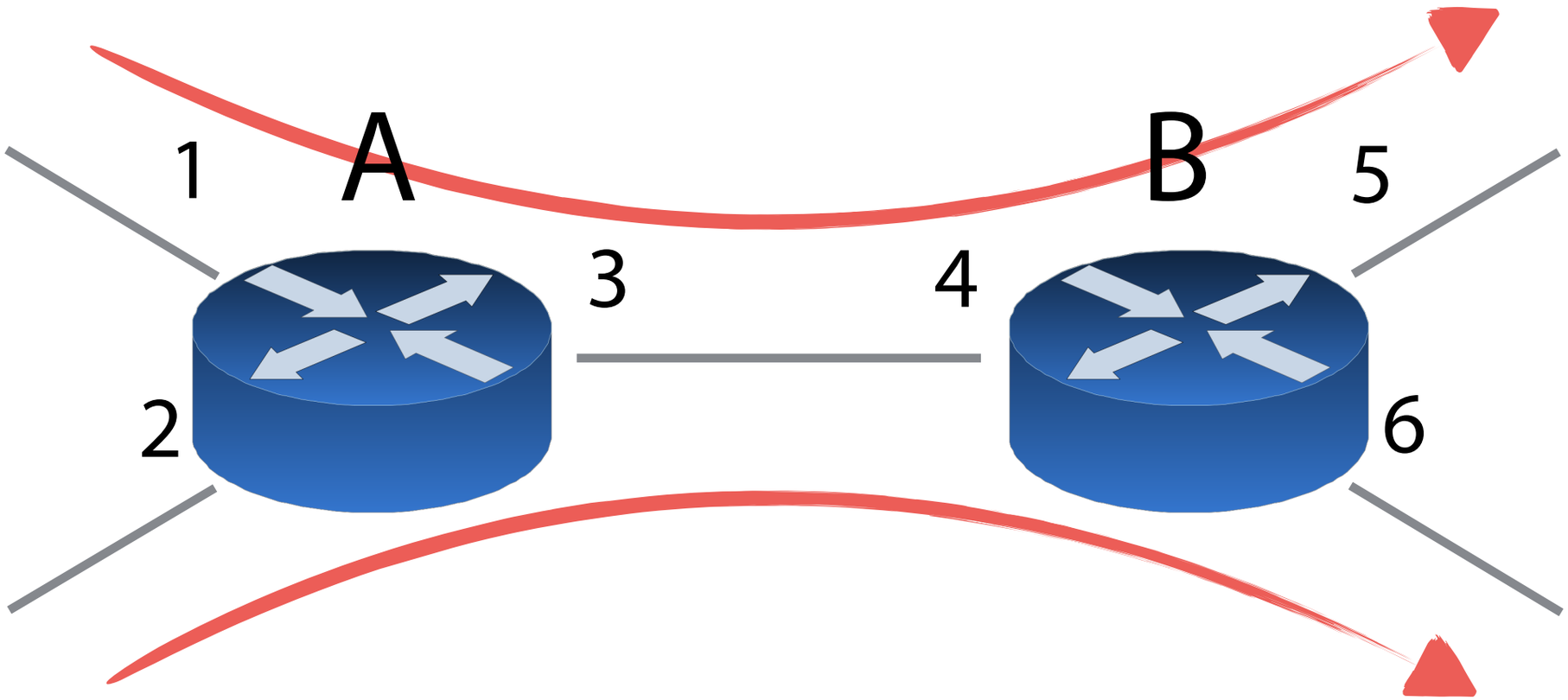
Local Program



port=1; tag:=1; port:=3
+
port=2; tag:=2; port:=3

tag=1; port:=5
+
tag=2; port:=6

Local Program

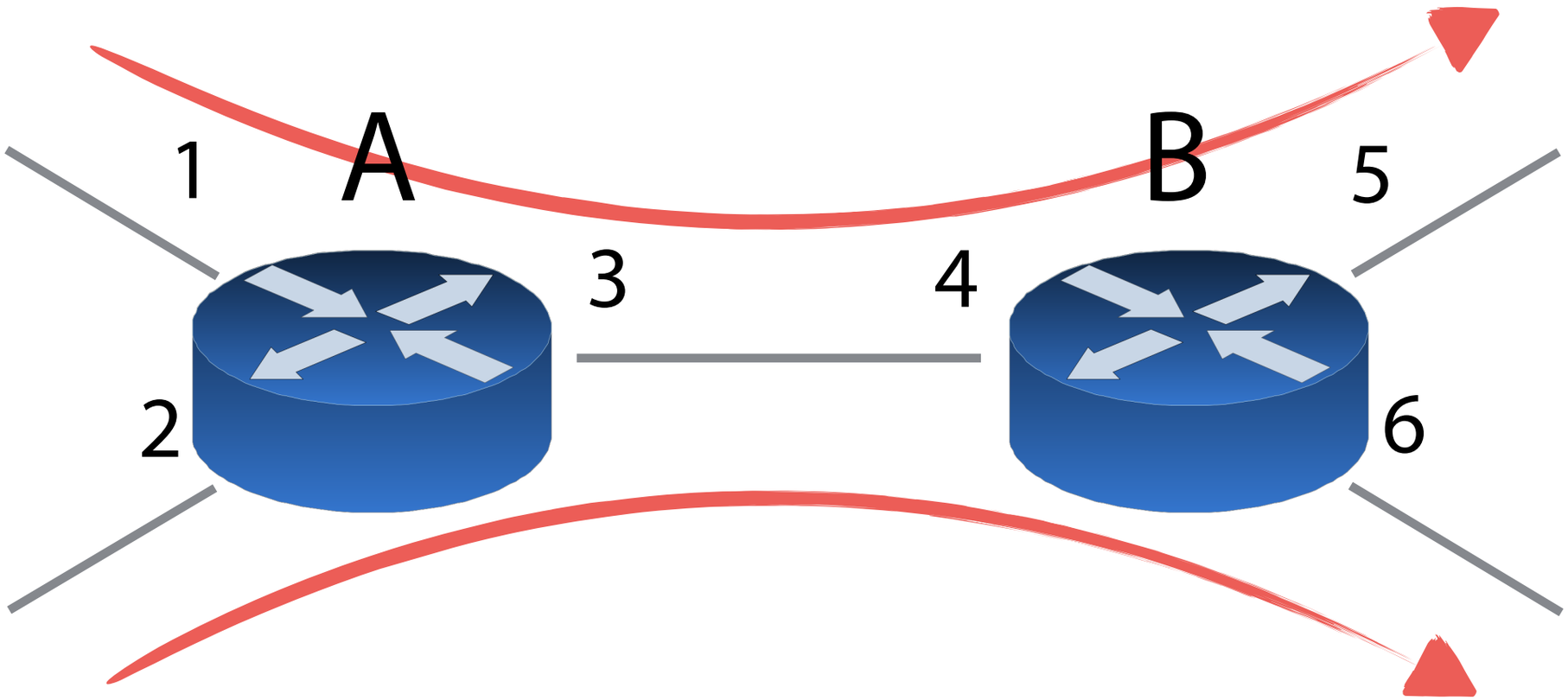



```
port=1; tag:=1; port:=3
        +
port=2; tag:=2; port:=3
```

```
tag=1; port:=5
        +
tag=2; port:=6
```

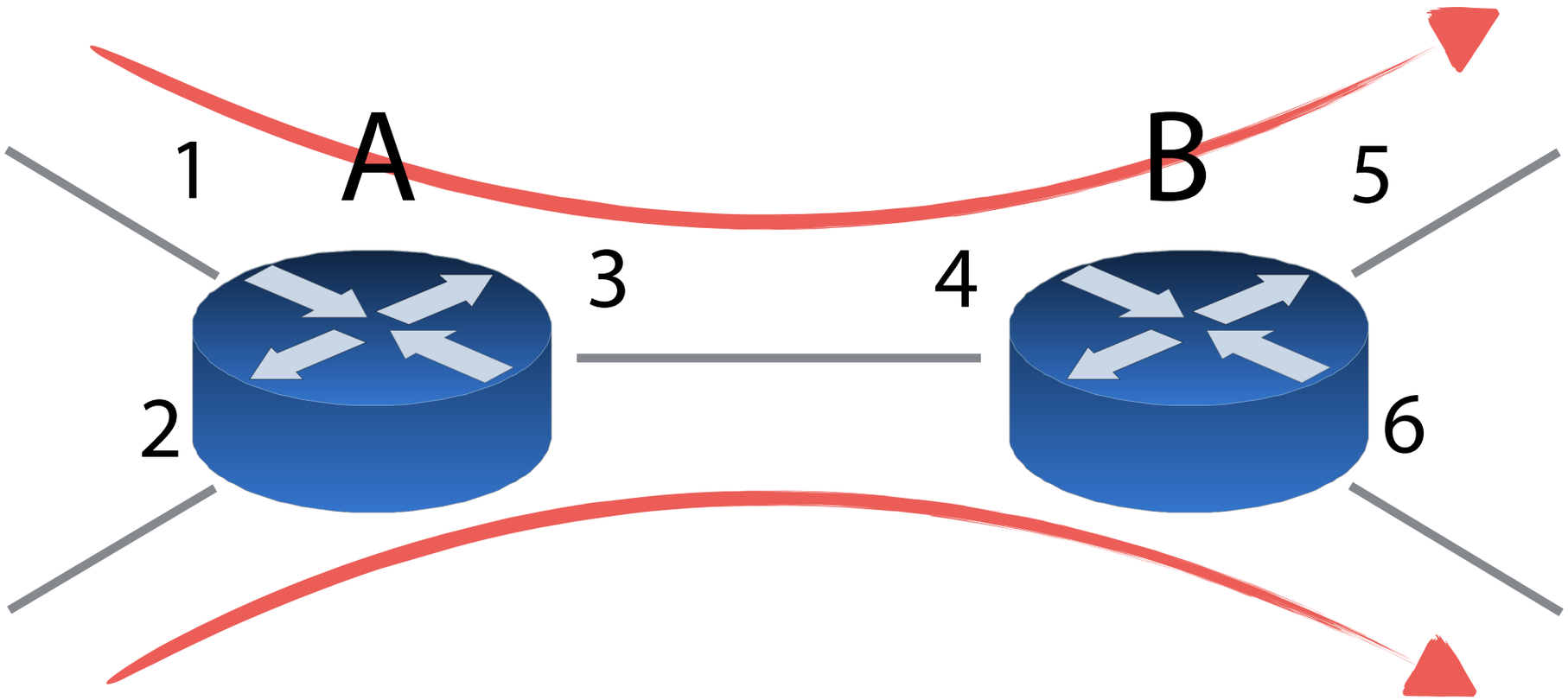
tedious for programmers… difficult to get right!

Global Program



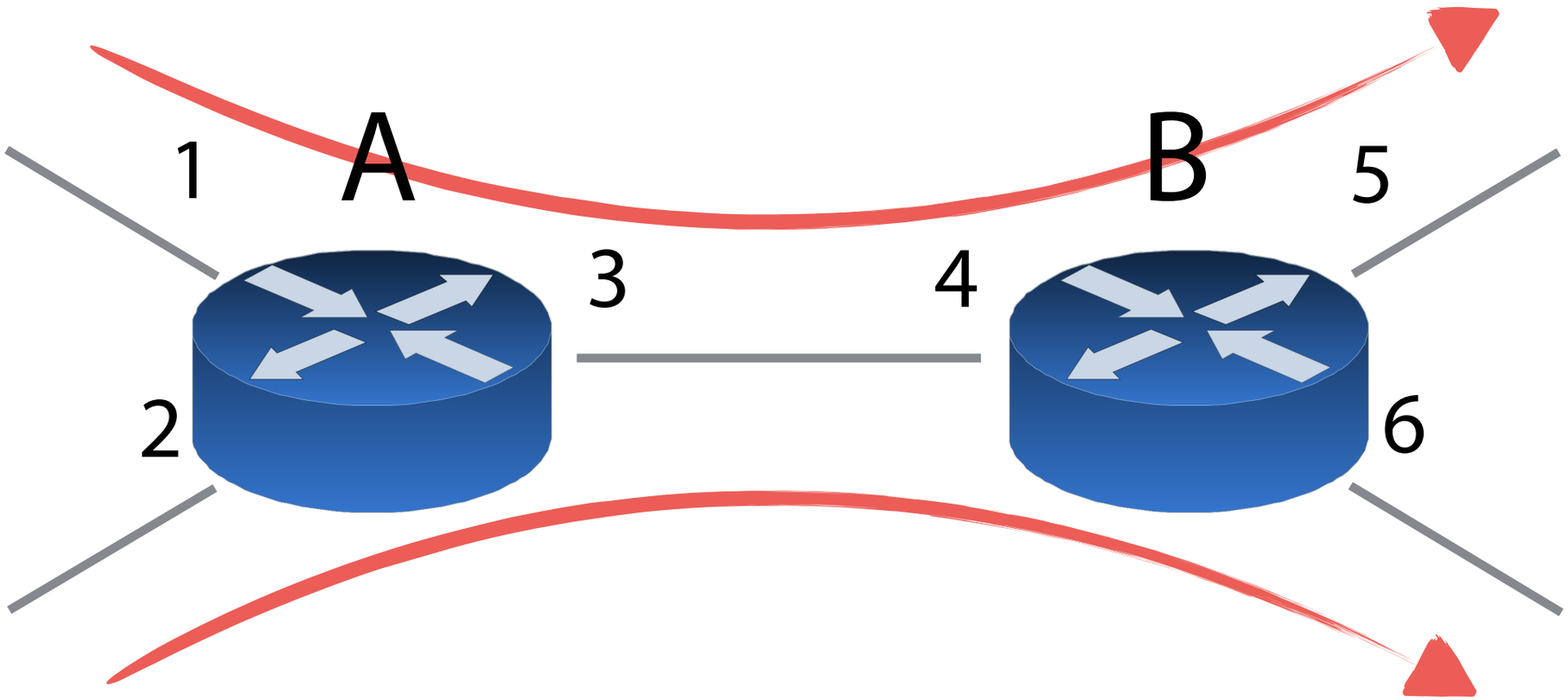
pol

Global Program



port=1**; A→B;** port**:=**5
+
port=2**; A→B;** port**:=**6

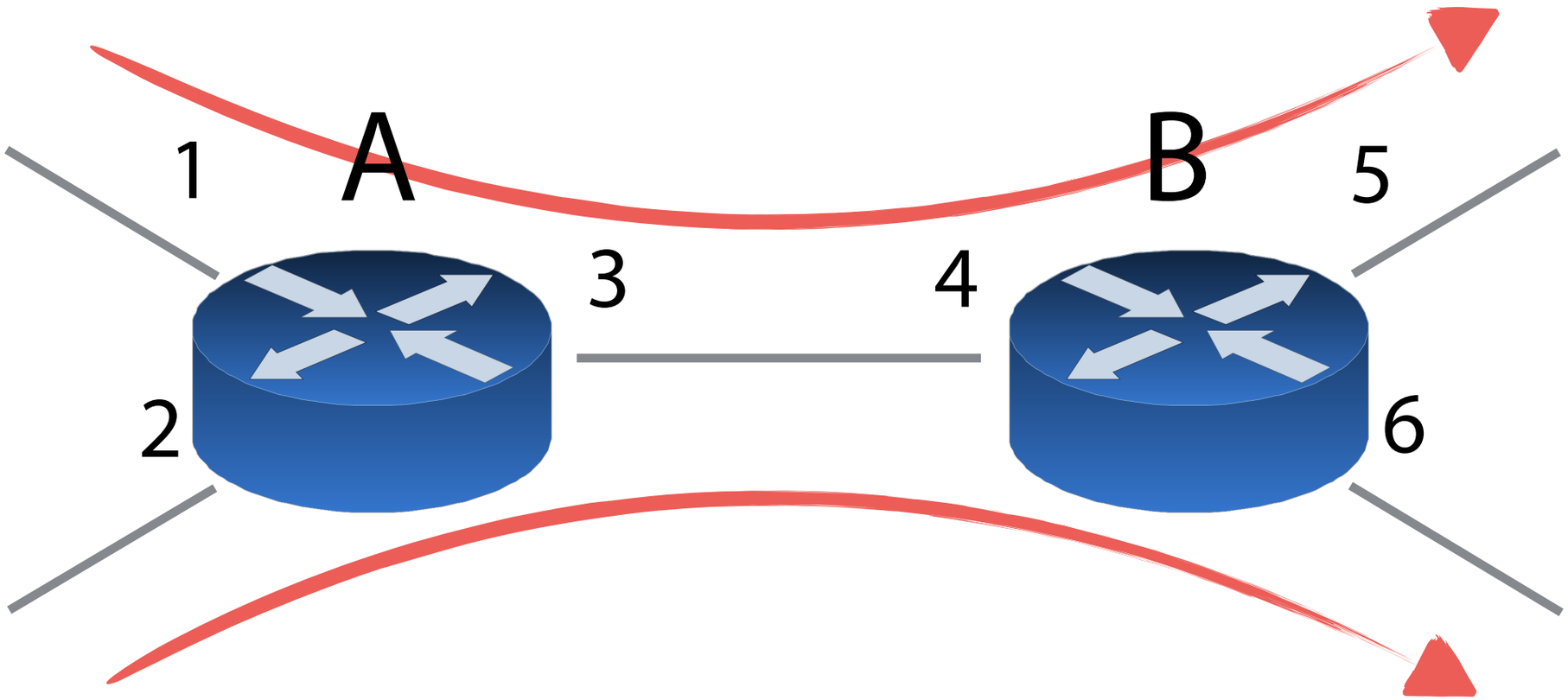
Global Program



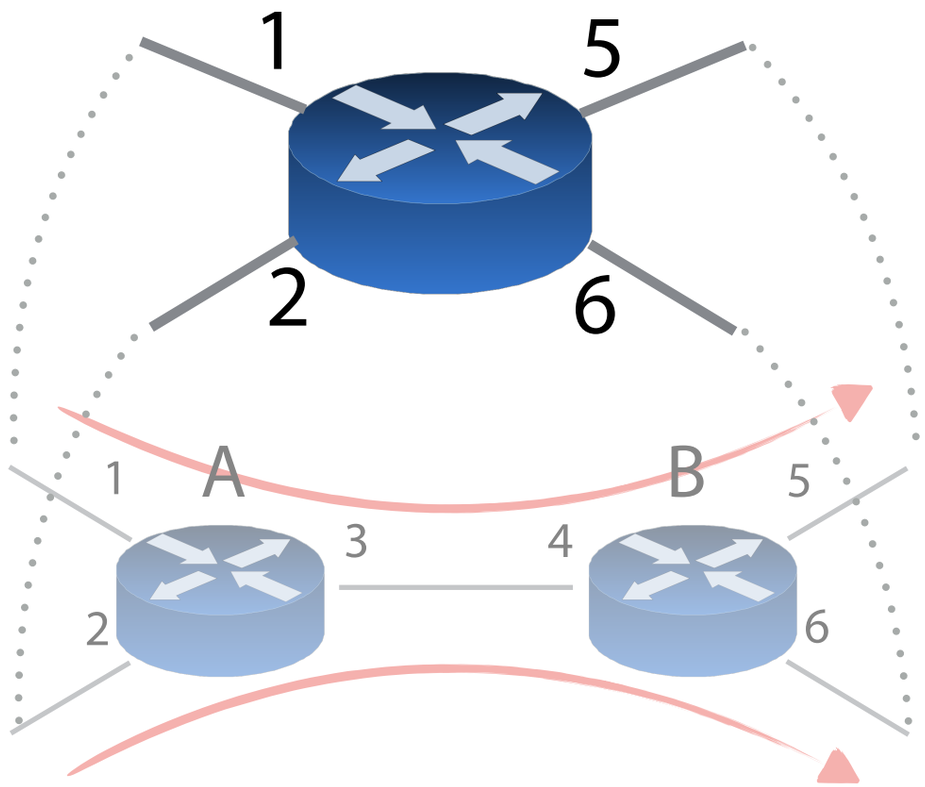
port=1; **A→B;** port**:=**5
+
port=2; **A→B;** port**:=**6

simple and elegant!

Virtual Program

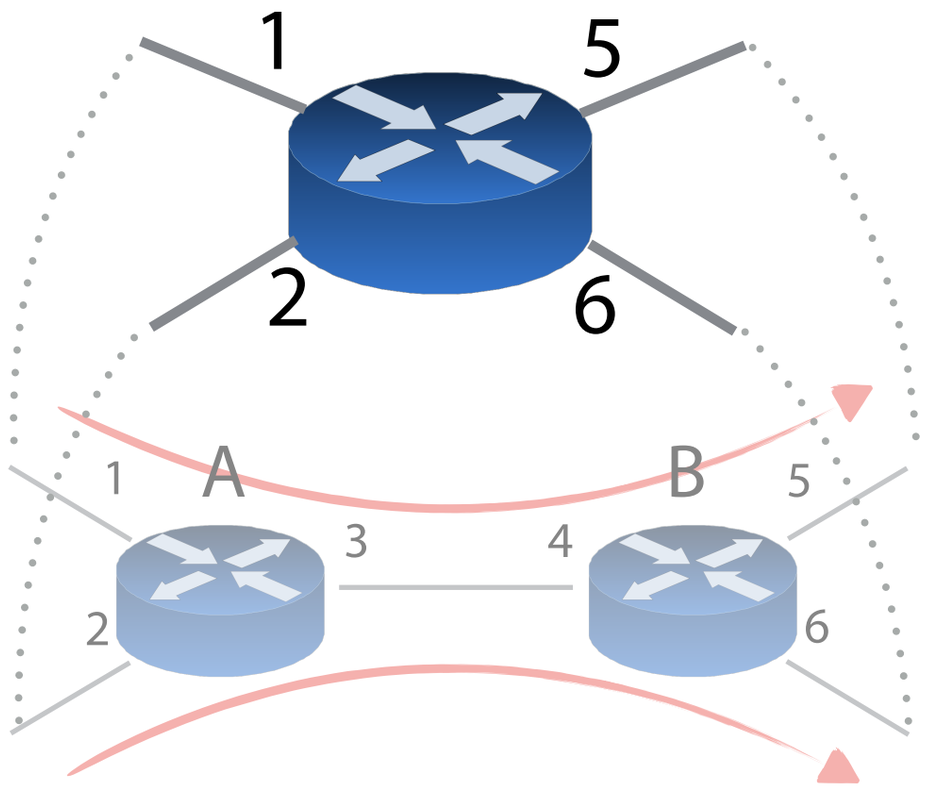


Virtual Program



virtual "big switch"

Virtual Program



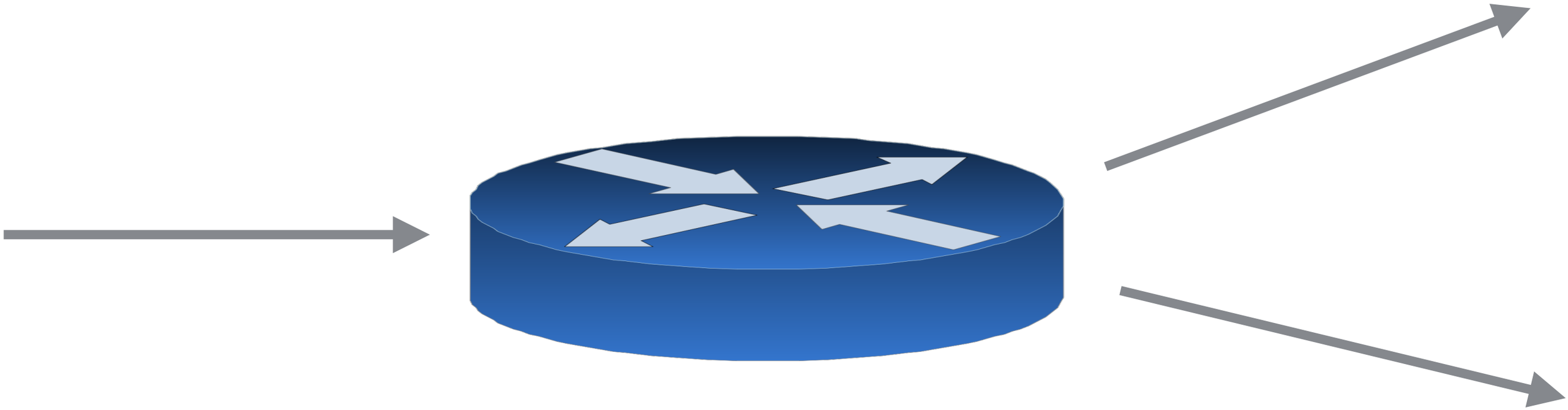
virtual "big switch"

port=1; port:=5
+
port=2; port:=6

even simpler!

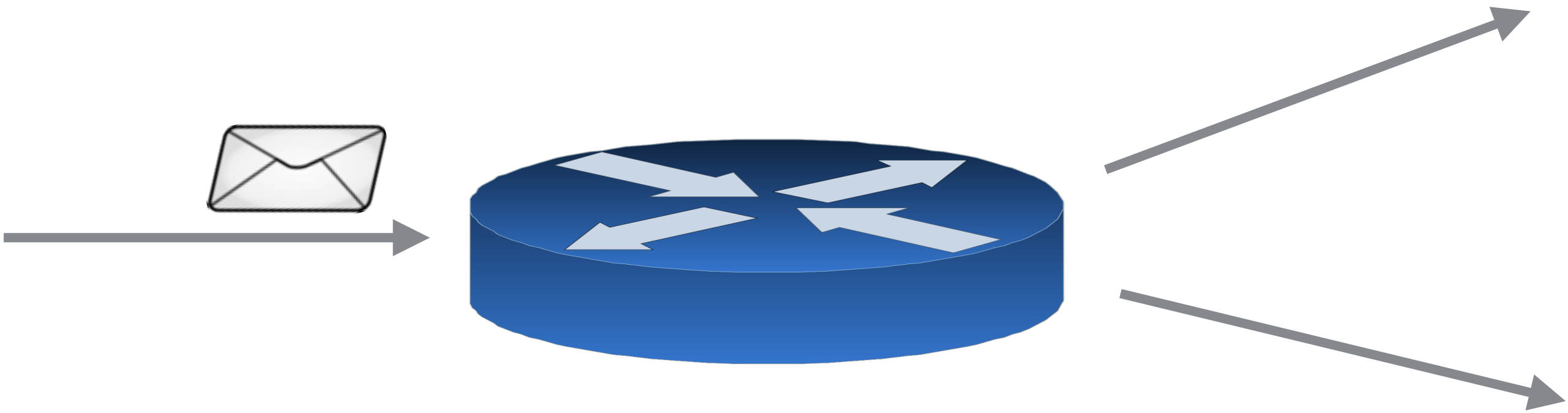
A Functional Forwarding Model

Model



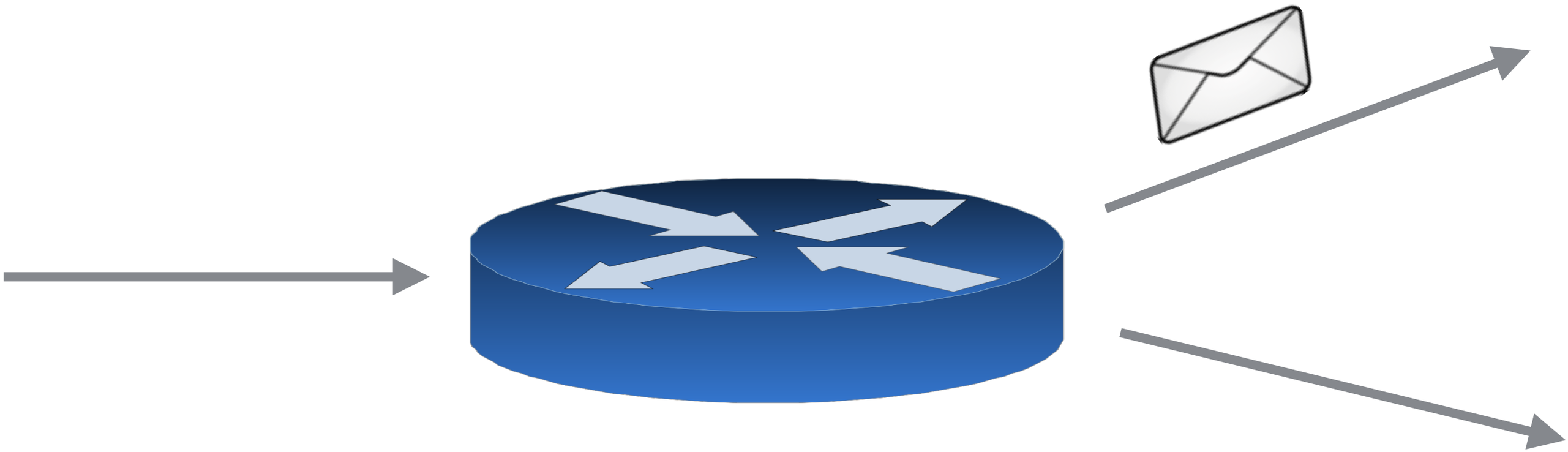
Packets → Packets

Model



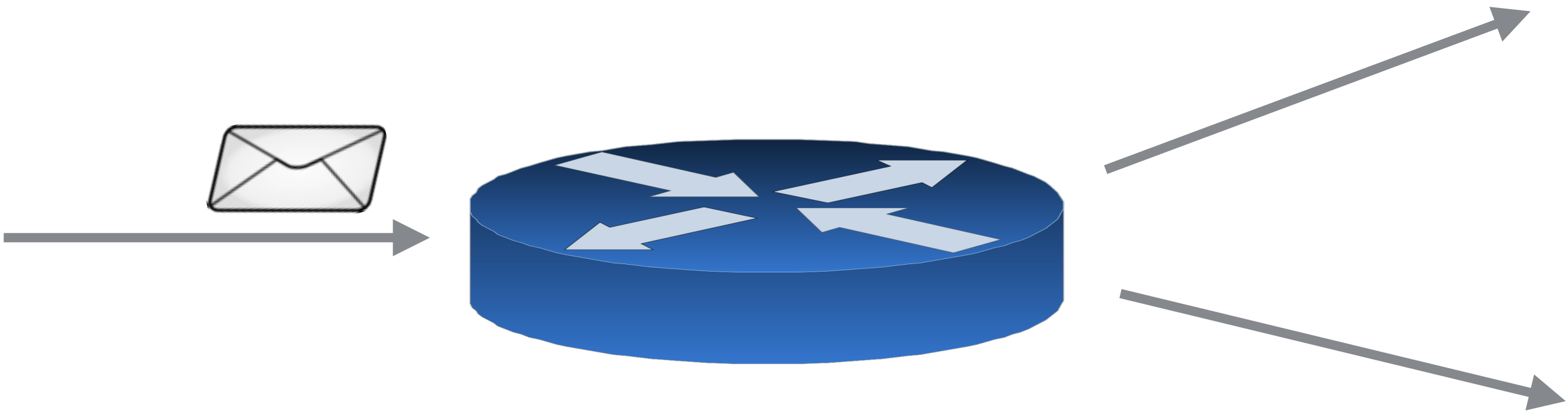
Packets → Packets

Model



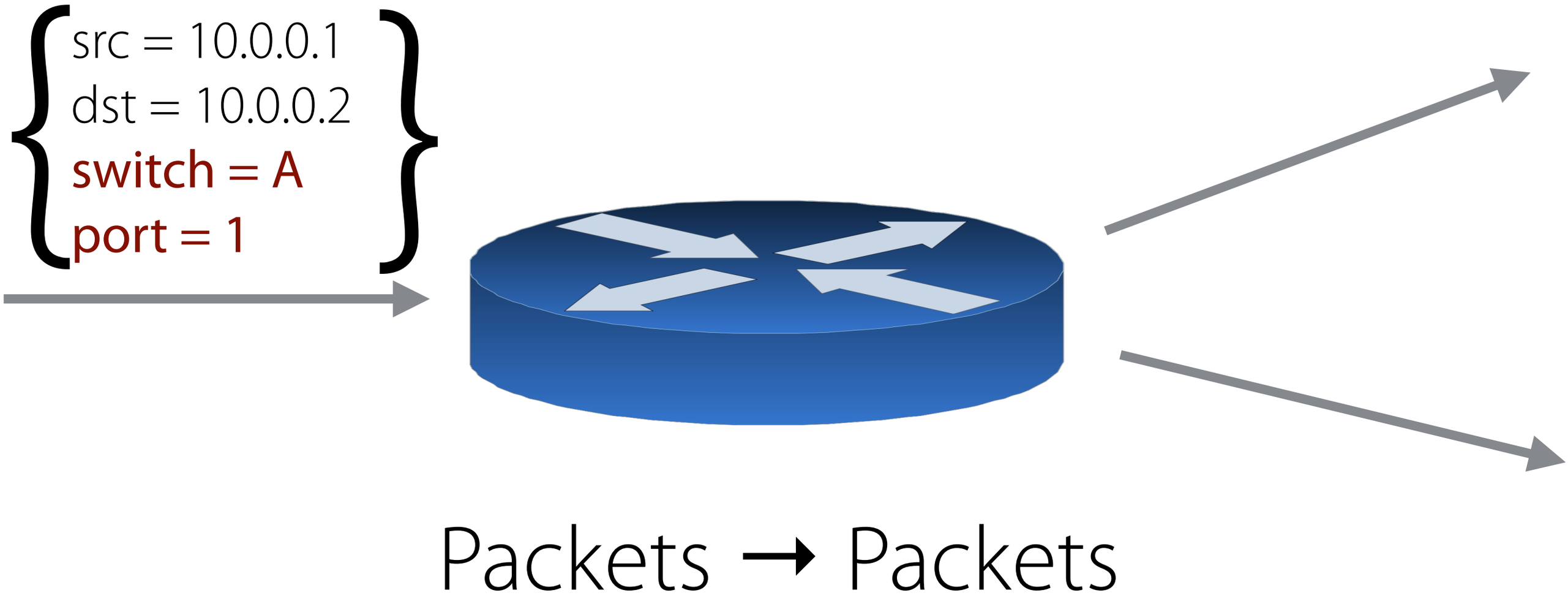
Packets → Packets

Model



Packets → Packets

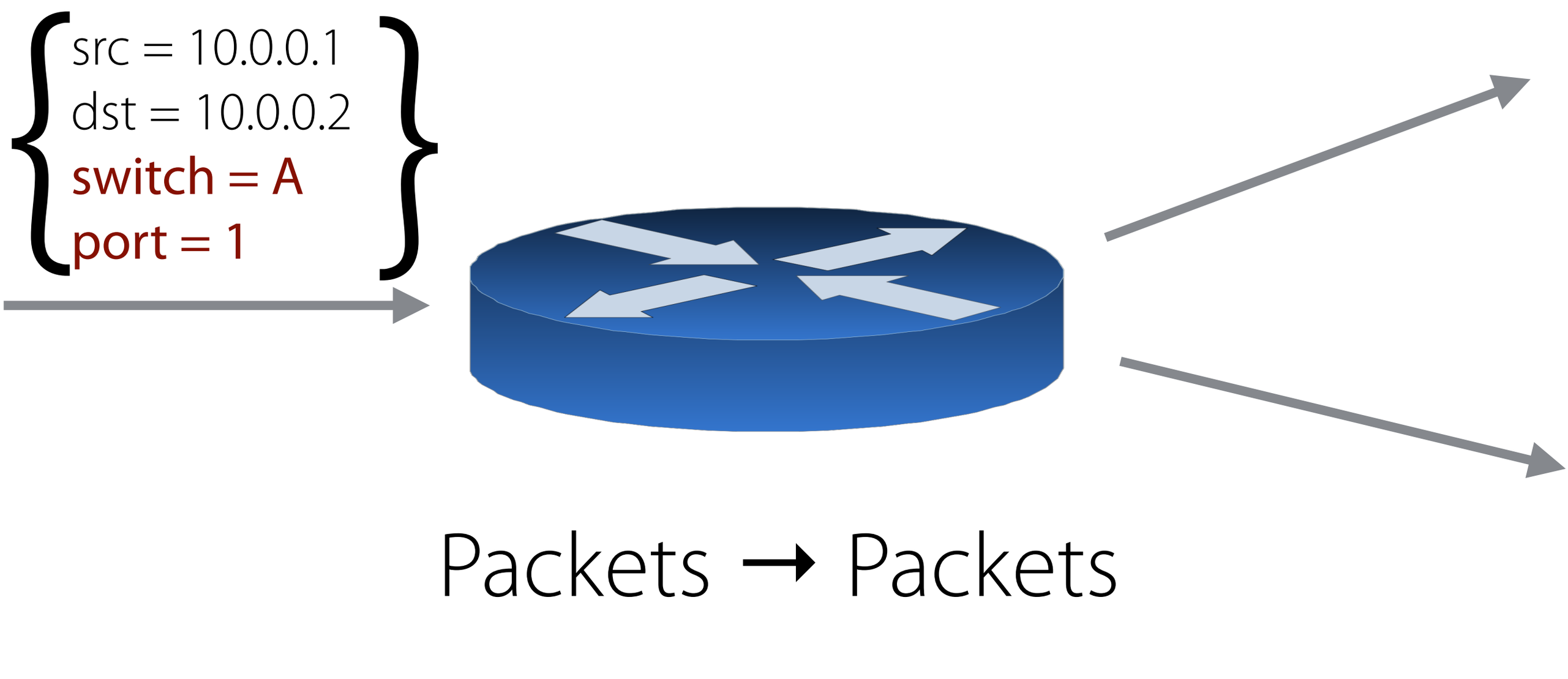
Model



Packets → Packets

Packet Manipulation

- packet filtering
- packet modification



Packets → Packets

Packet Manipulation

- **packet filtering**
- packet modification

Packets → Packets

Packet Manipulation

- **packet filtering**
- packet modification

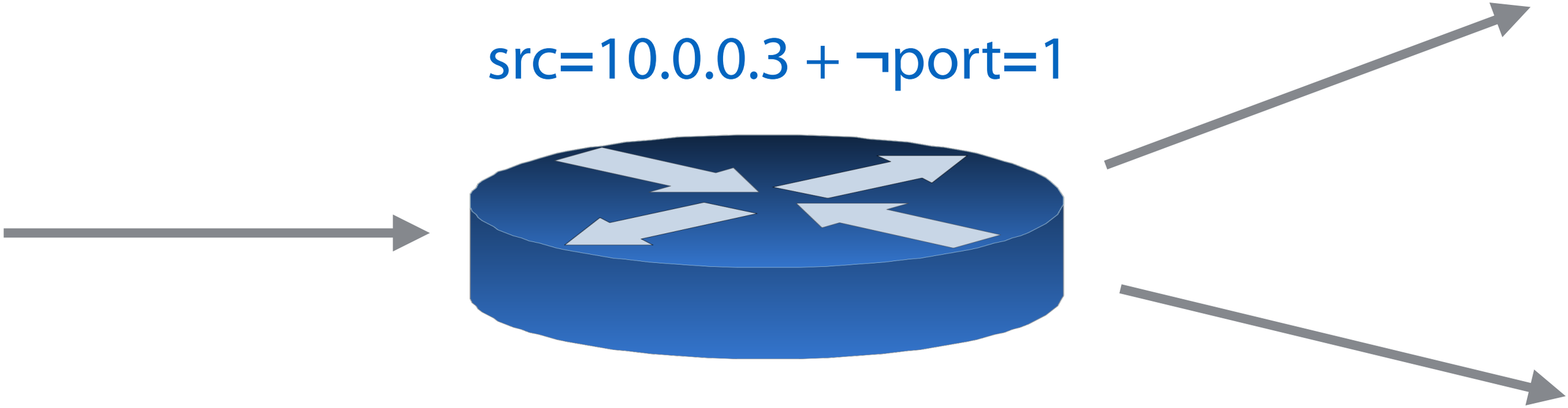
{ src = 10.0.0.1
dst = 10.0.0.2
switch = A
port = 1 }

src=10.0.0.3 + ¬port=1

Packets → Packets

Packet Manipulation

- **packet filtering**
- packet modification



Packets → Packets

Packet Manipulation

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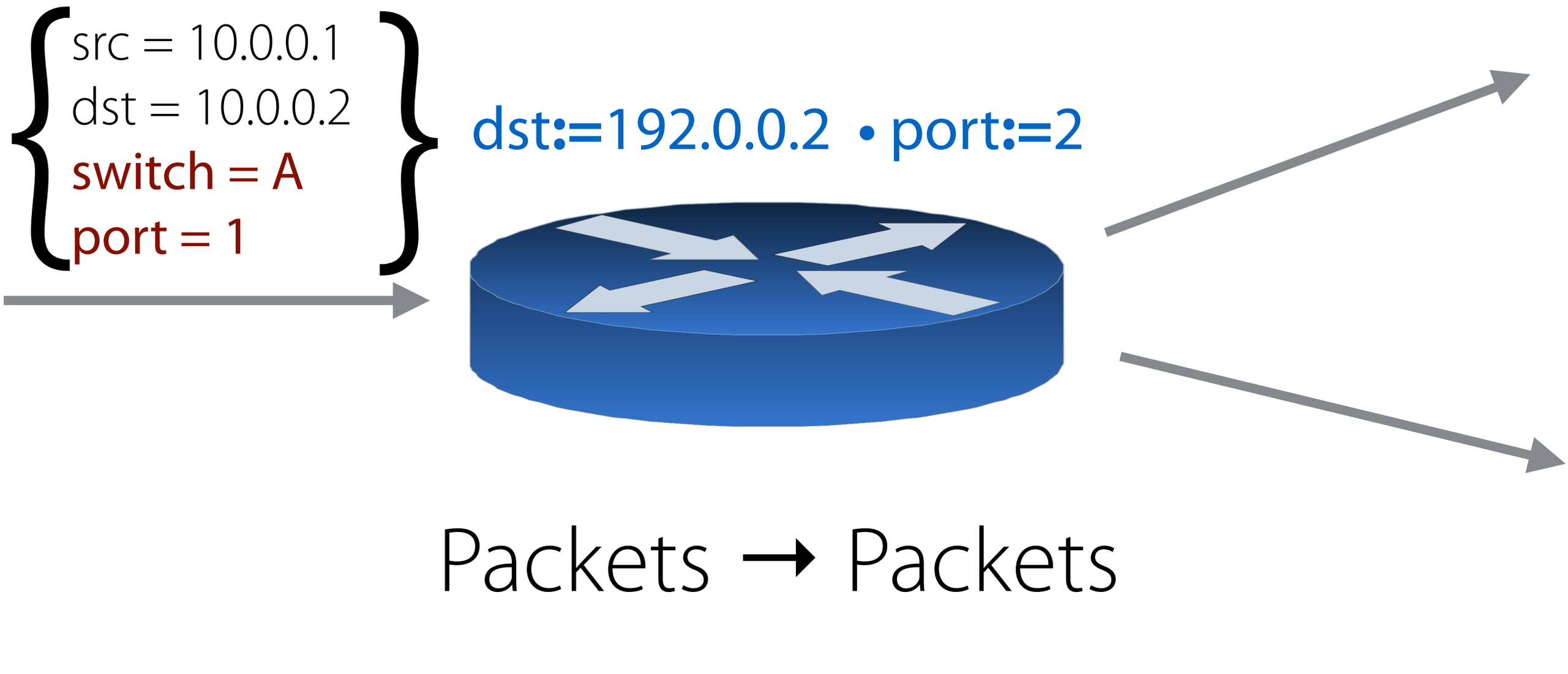

Packets → Packets

Packet Manipulation

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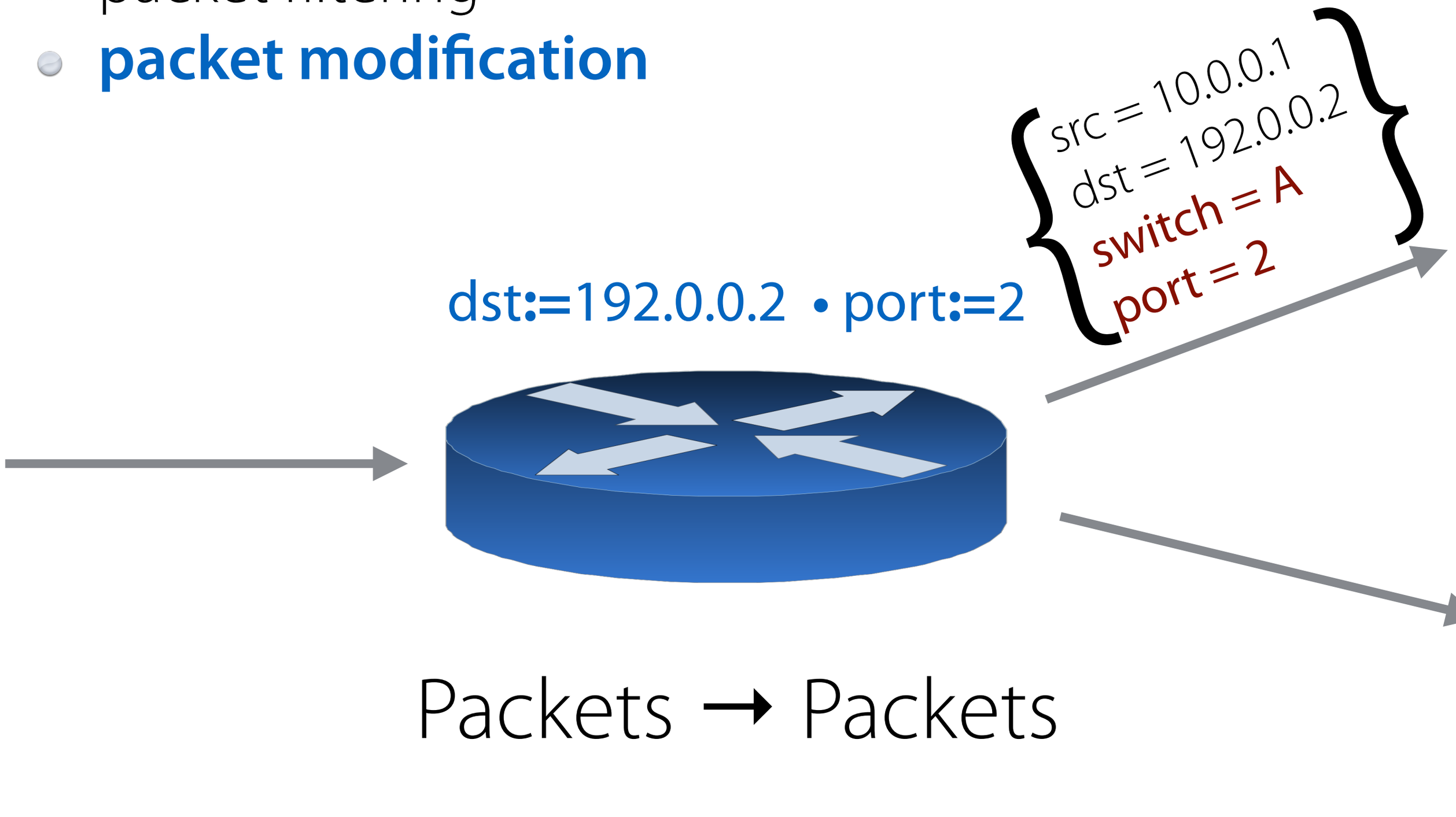

Packets → Packets

Topology

Topology

switch=A • pt=1 • switch:=B • pt:=2

Topology

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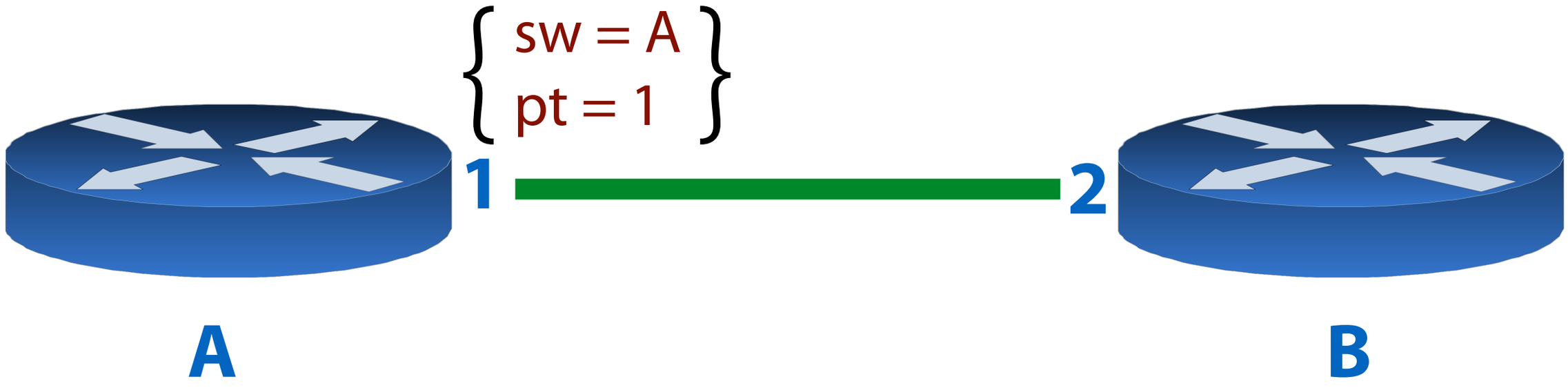
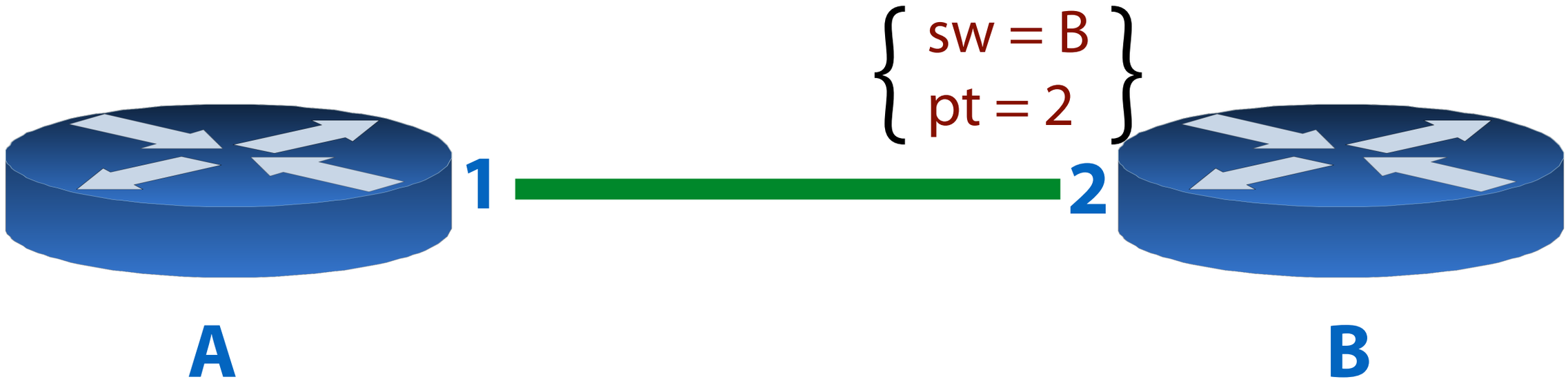


Topology

switch=A • pt=1 • switch:=B • pt:=2



Topology

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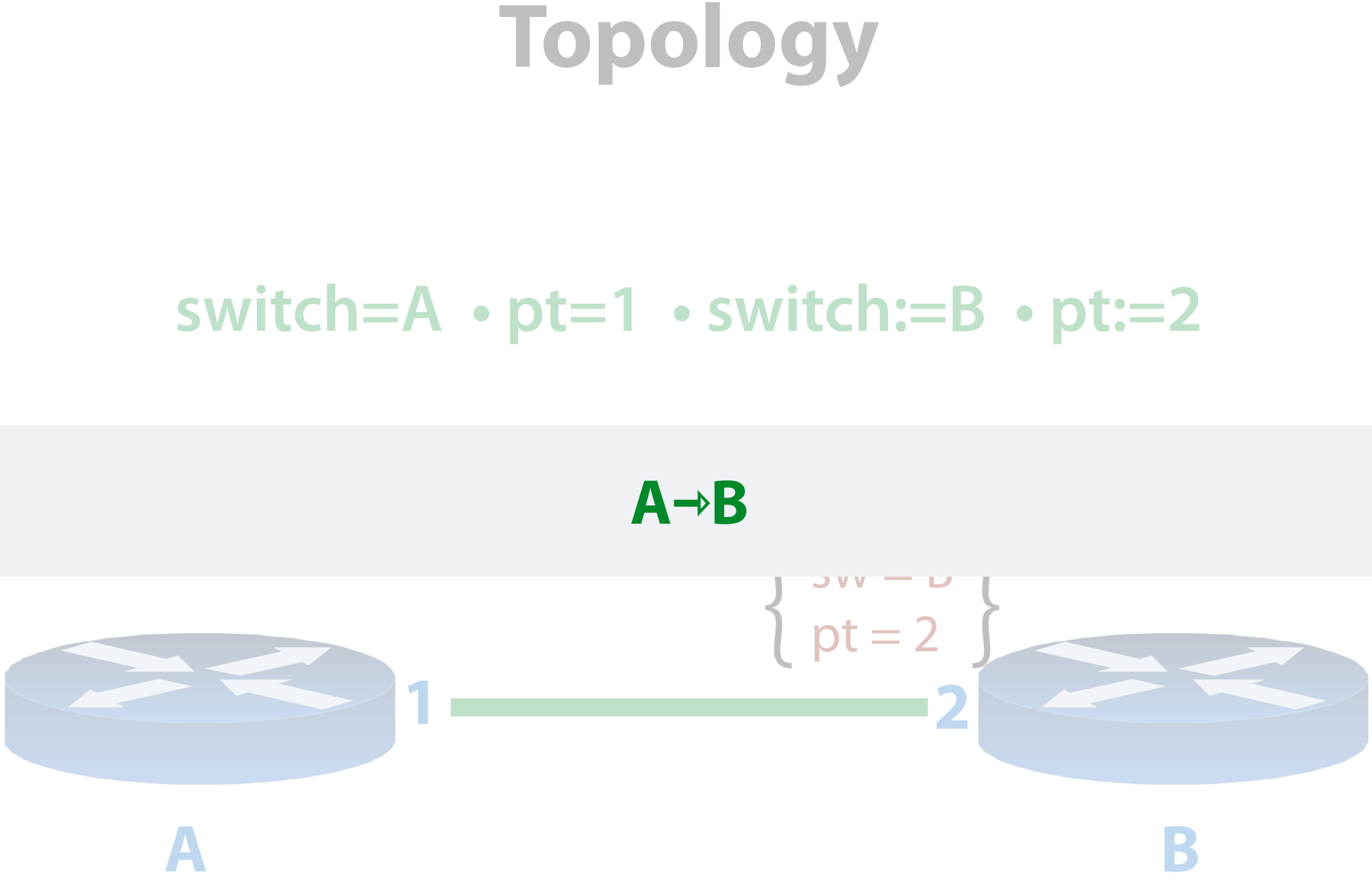
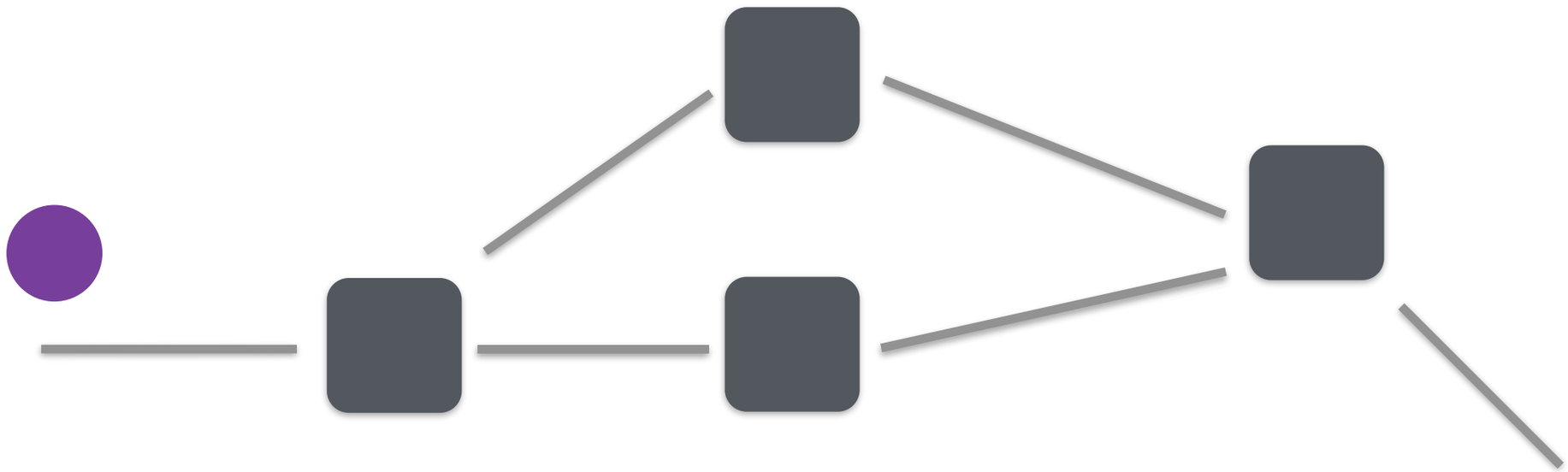
A→B



Iteration

(topology • switch)*



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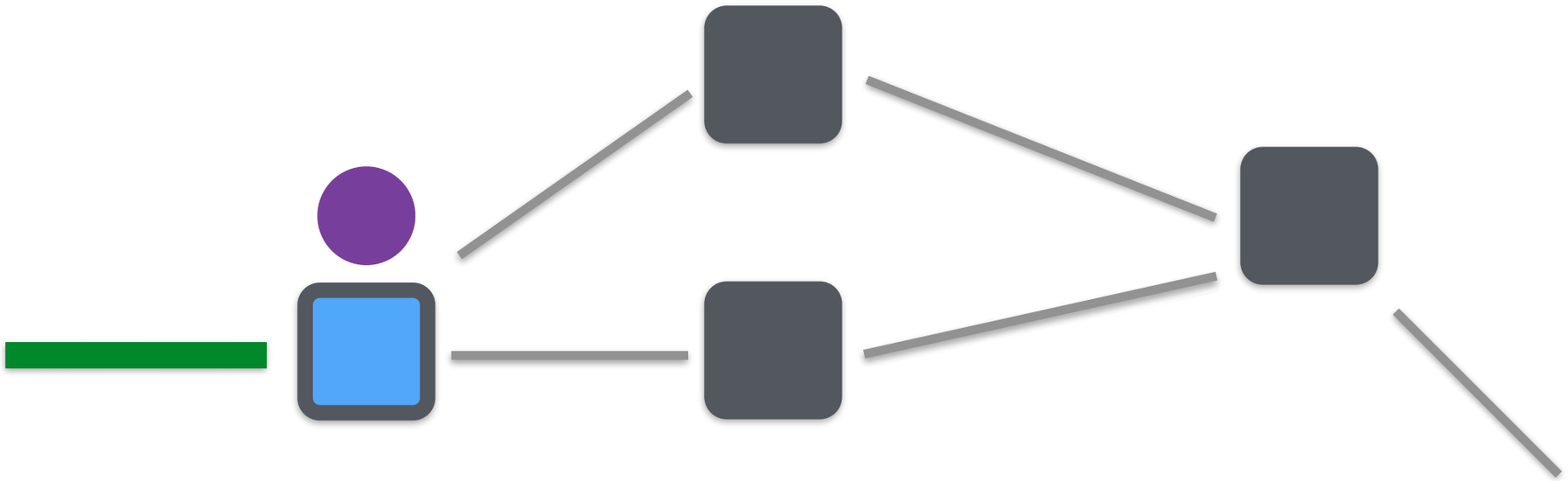

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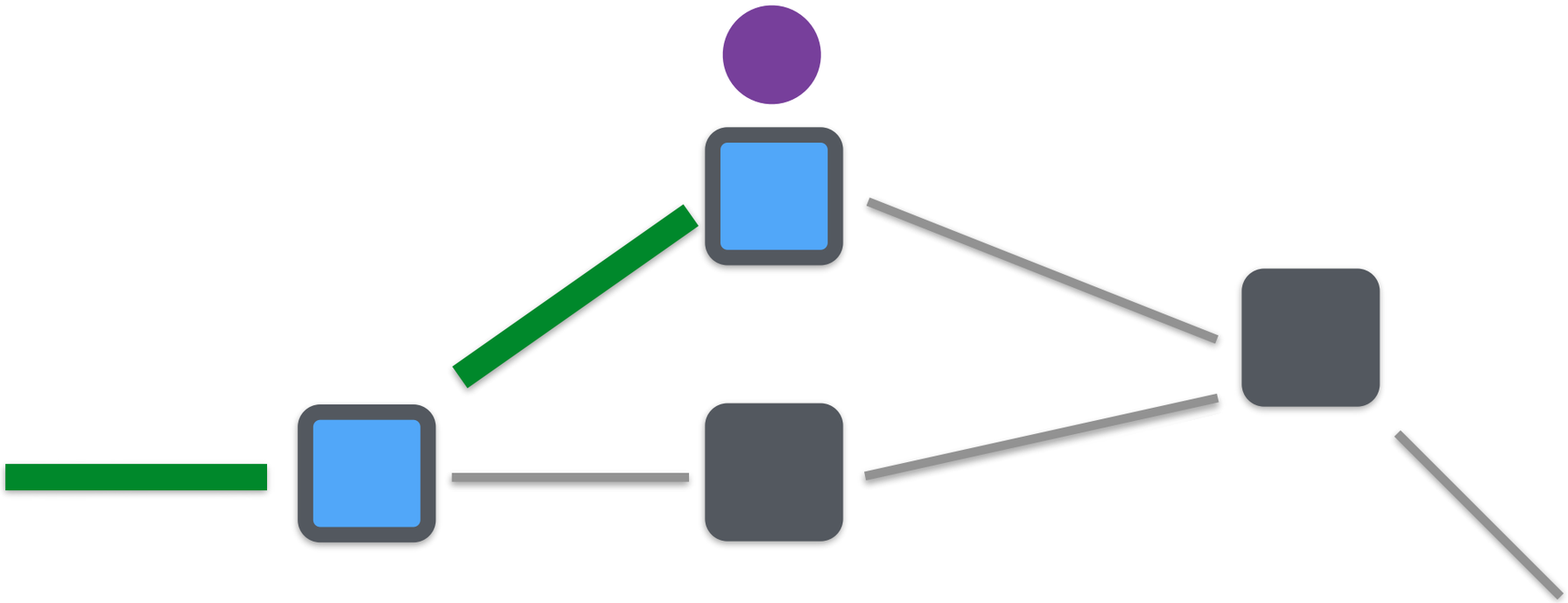

Iteration

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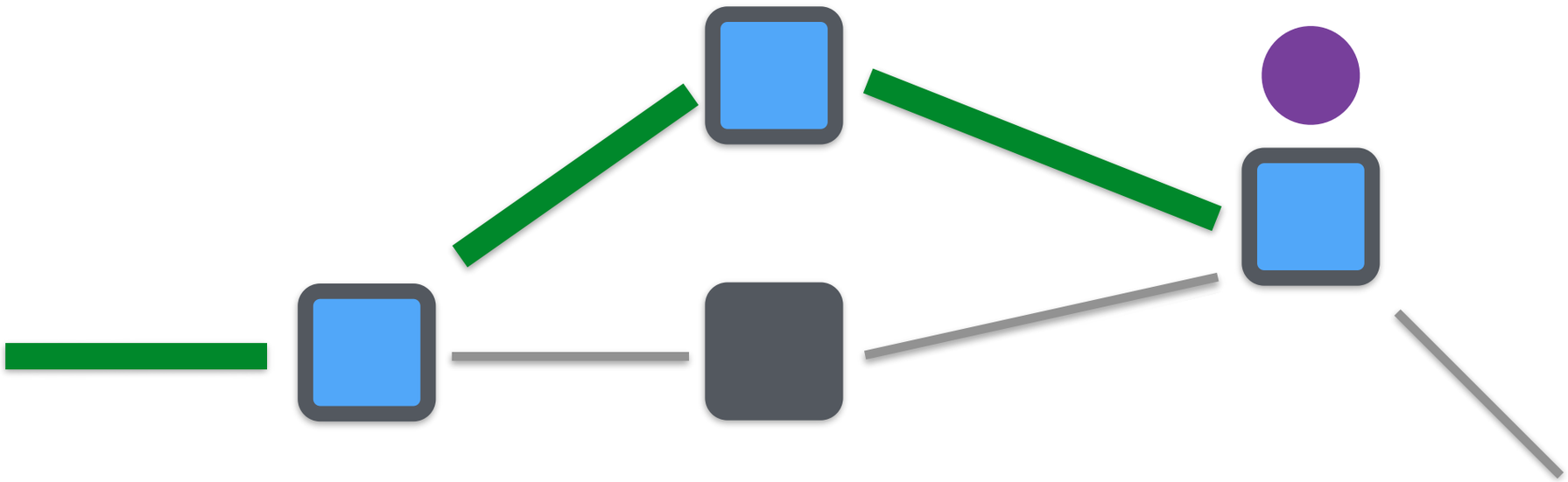

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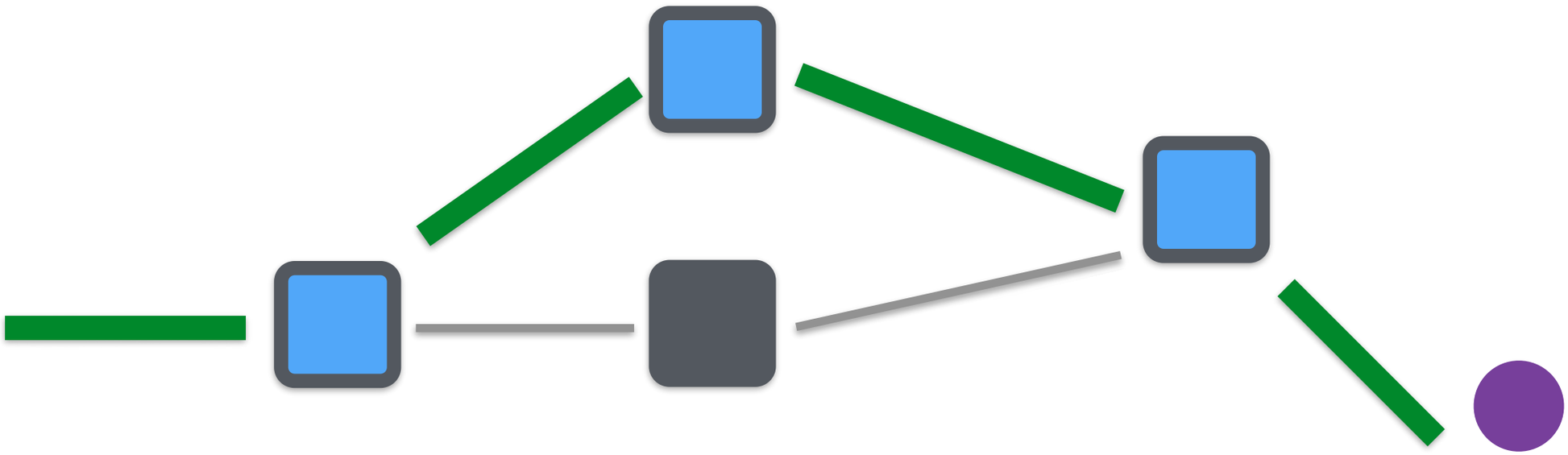

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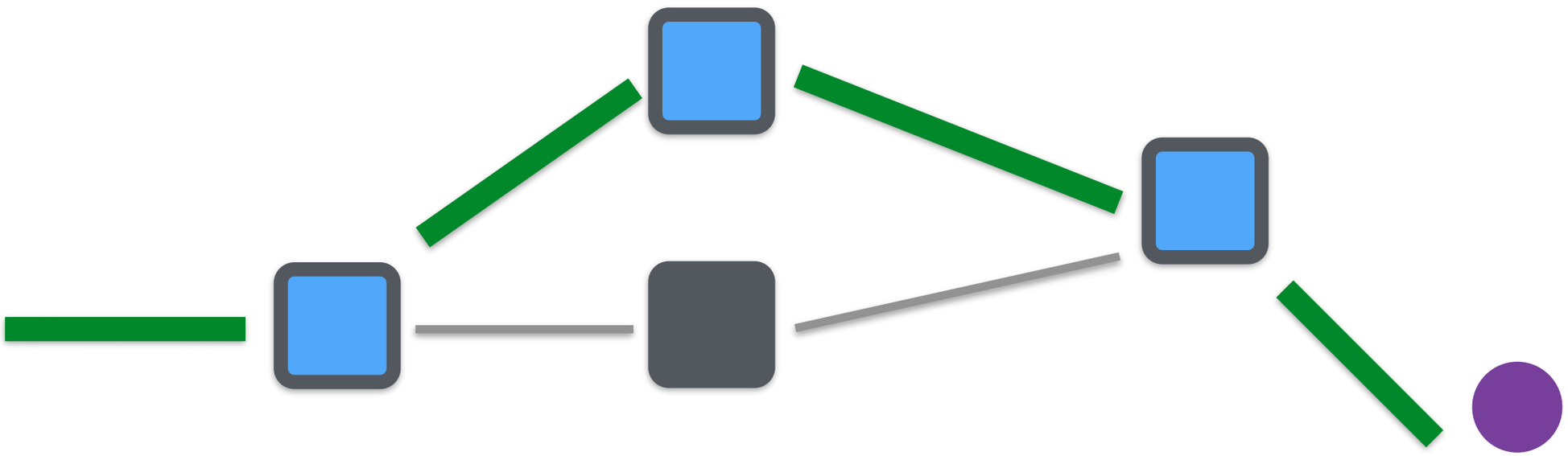
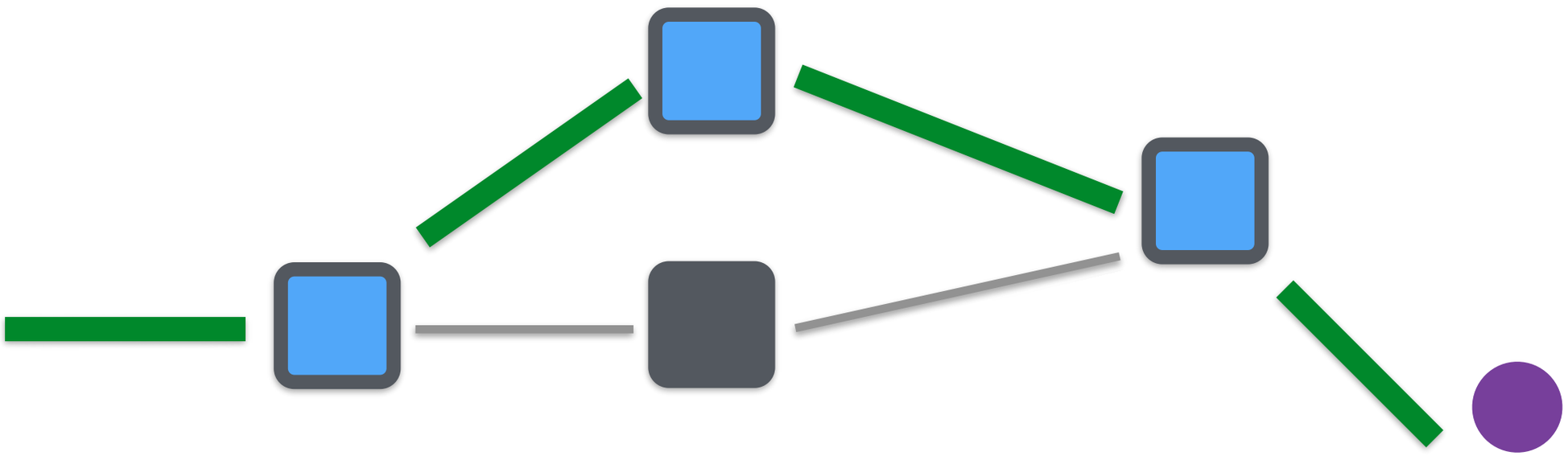


Packets → Packets

Iteration

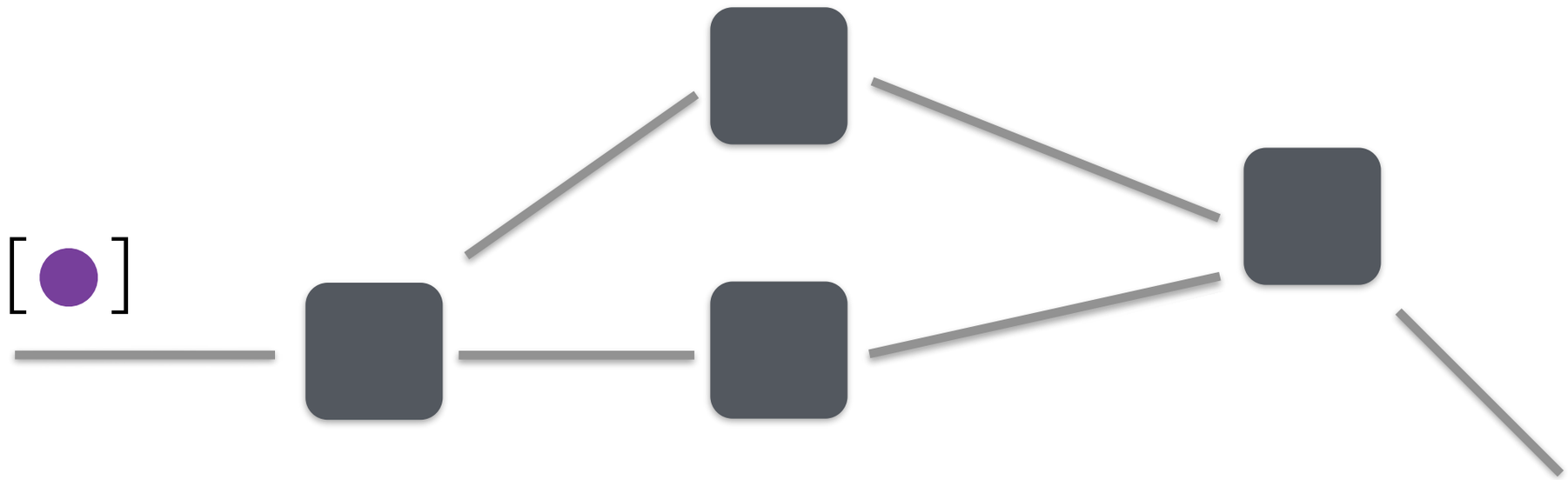
(topology • switch)*



~~Packets~~ ~~Packets~~

Iteration

(topology • switch)*

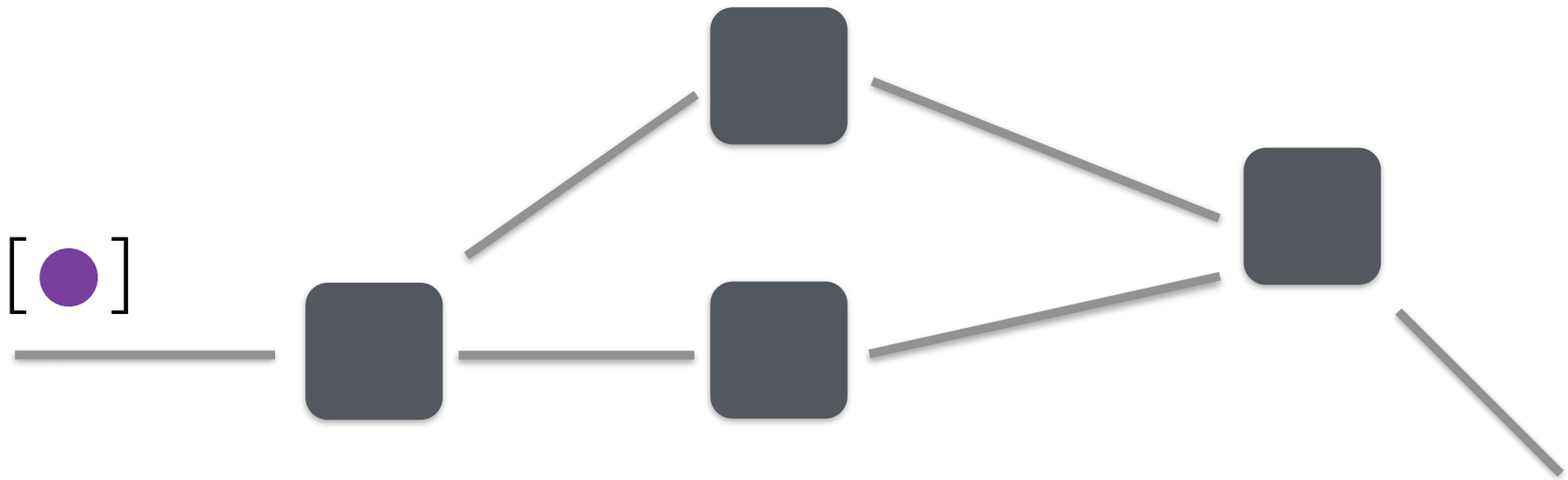


Packets → Packets

Histories → Histories

Iteration

(topology • switch)*



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Histories → Histories

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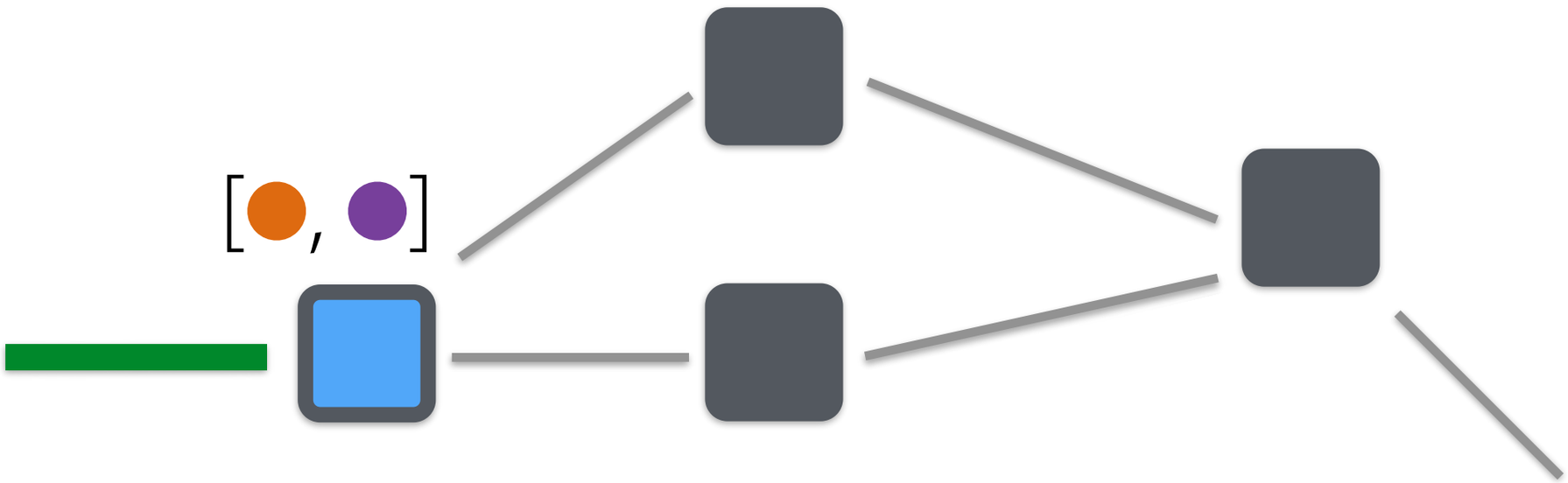

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Histories → Histories

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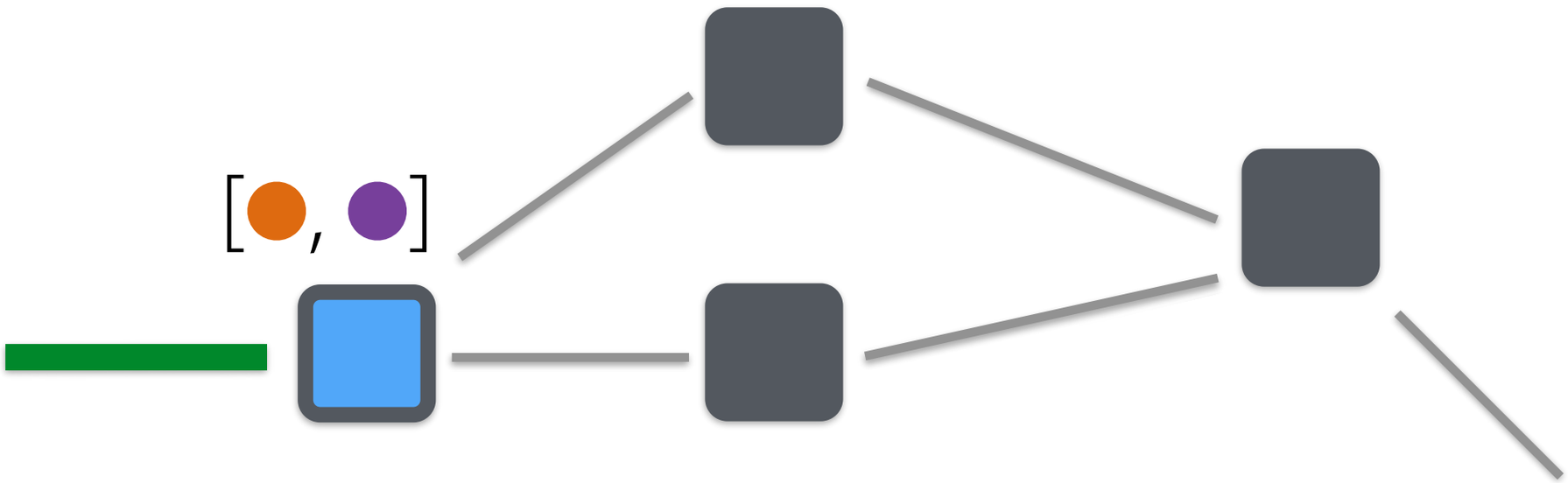
(topology • switch)*



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Histories → Histories

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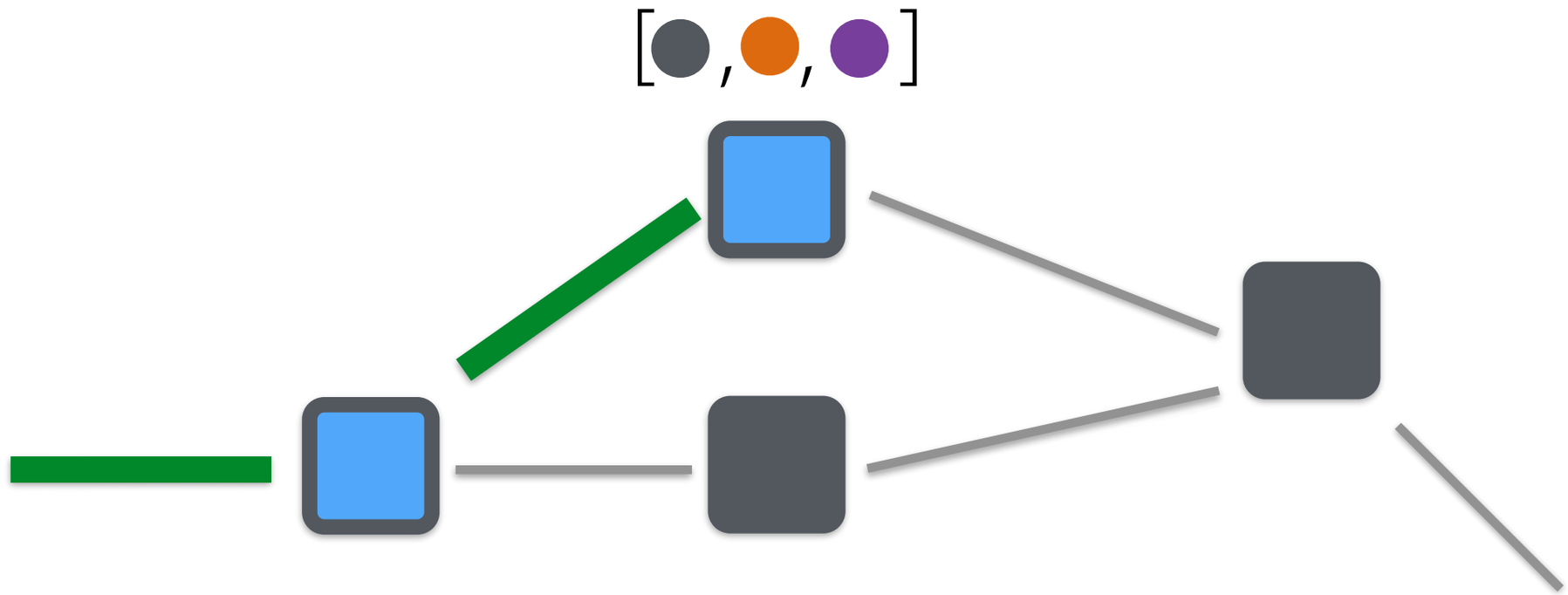

Packets → Packets

Histories → Histories

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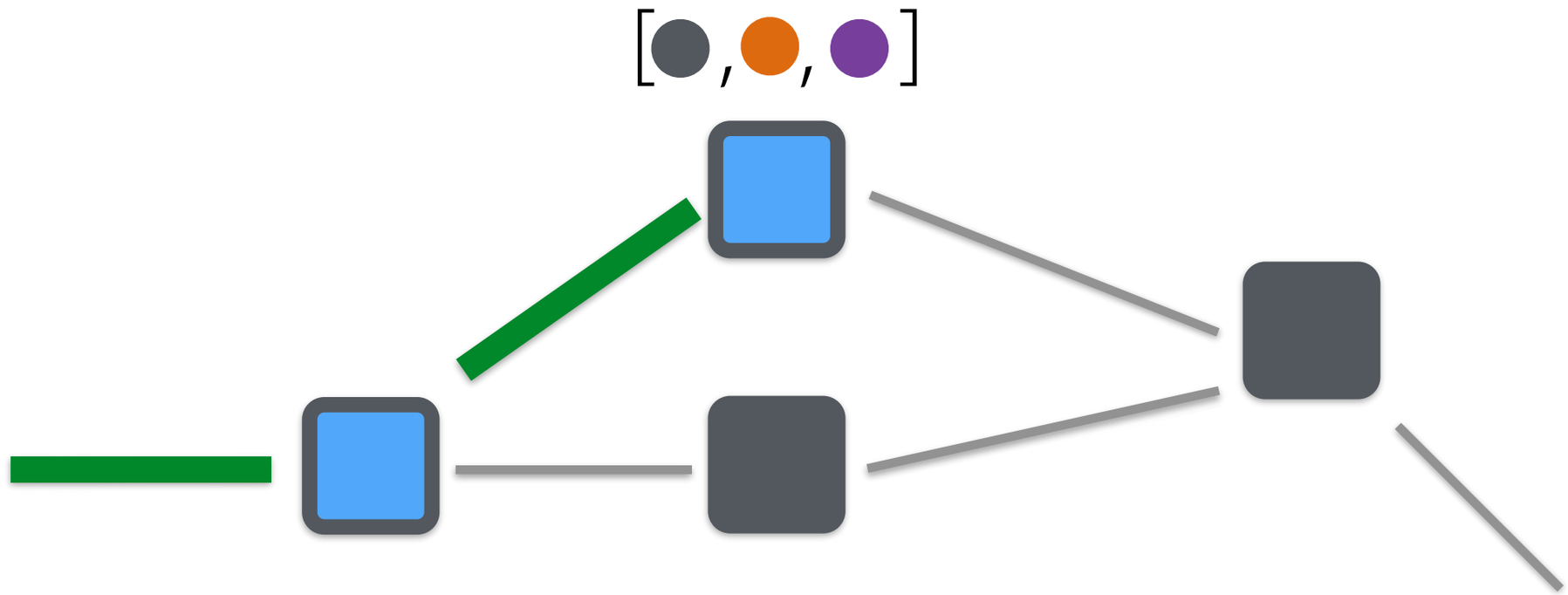

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Iteration

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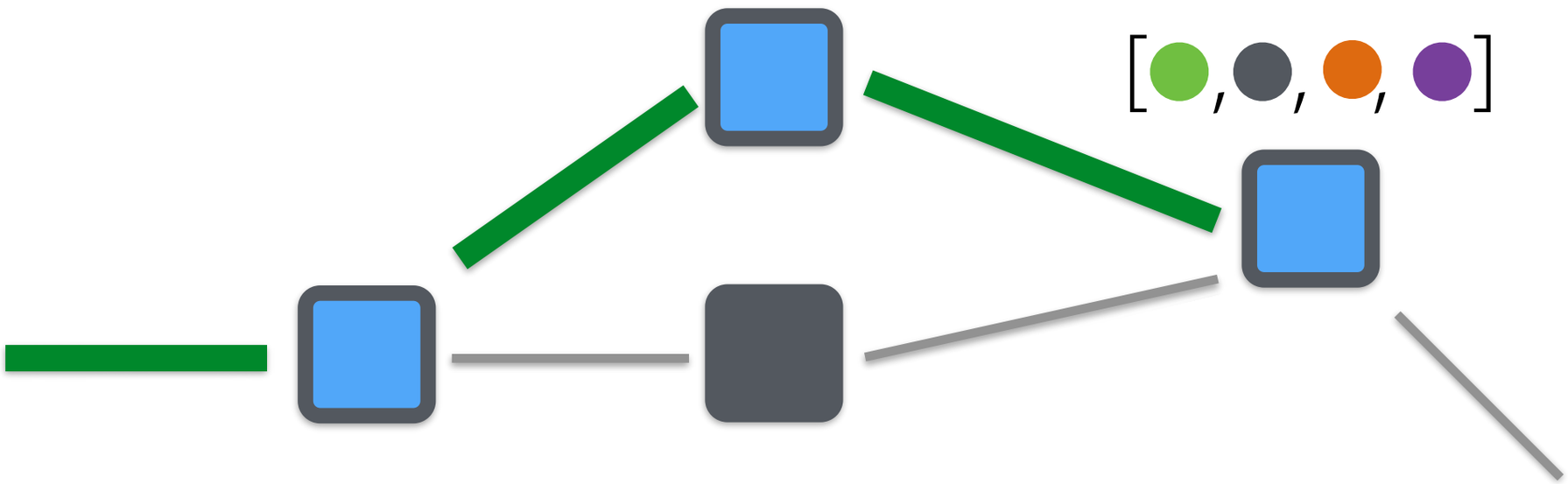

Packets → Packets

Histories → Histories

Iteration

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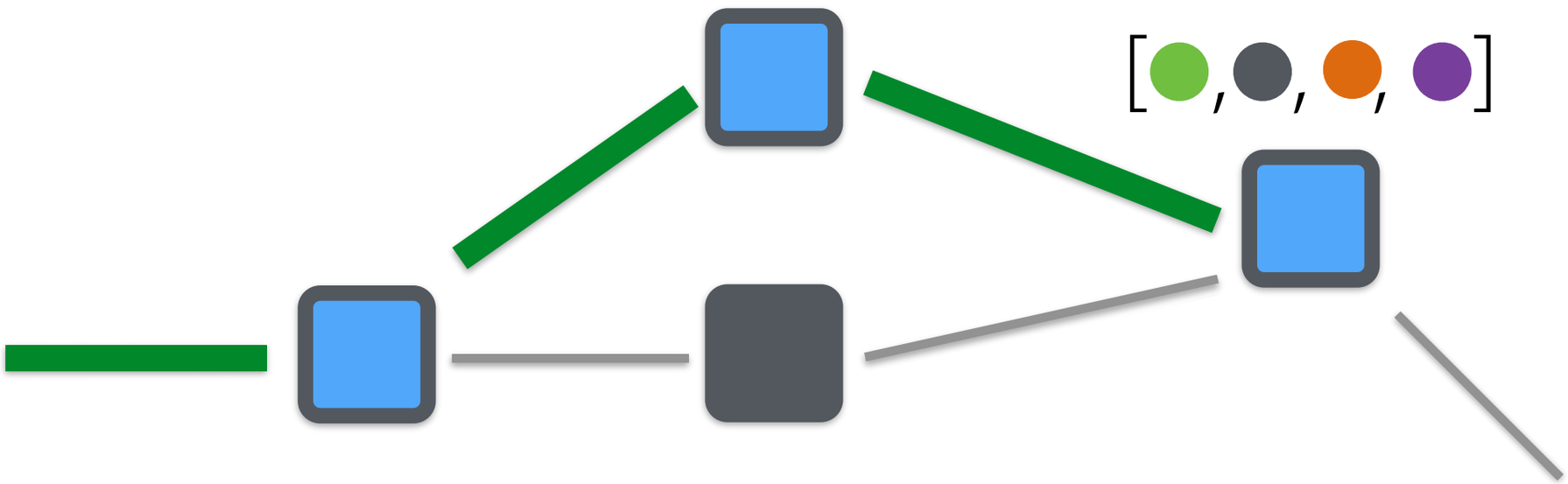

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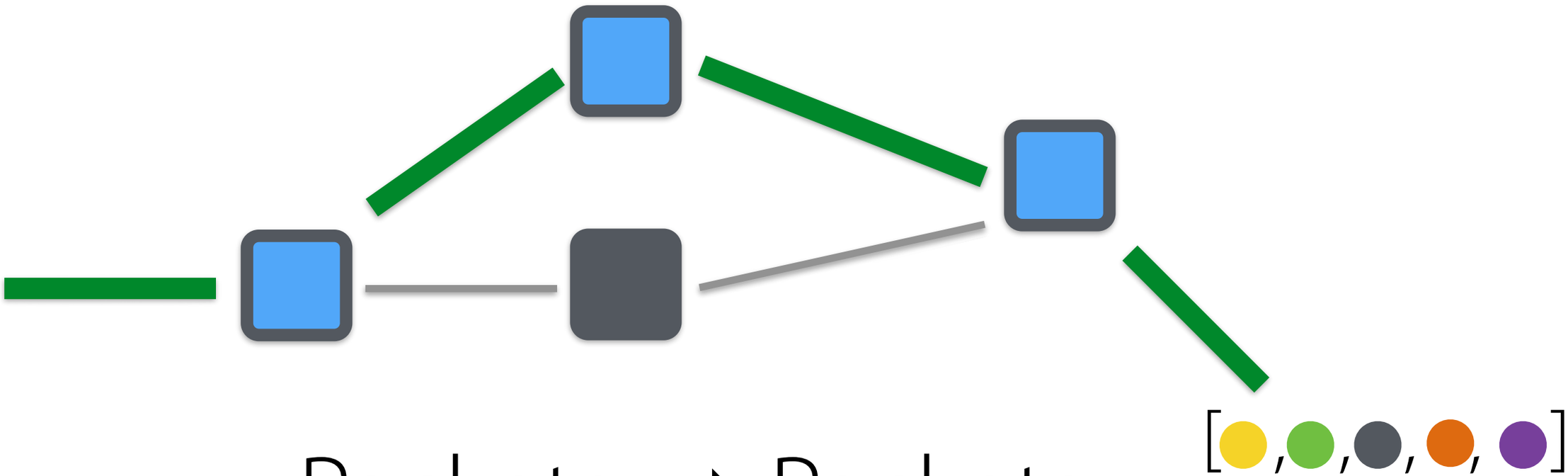

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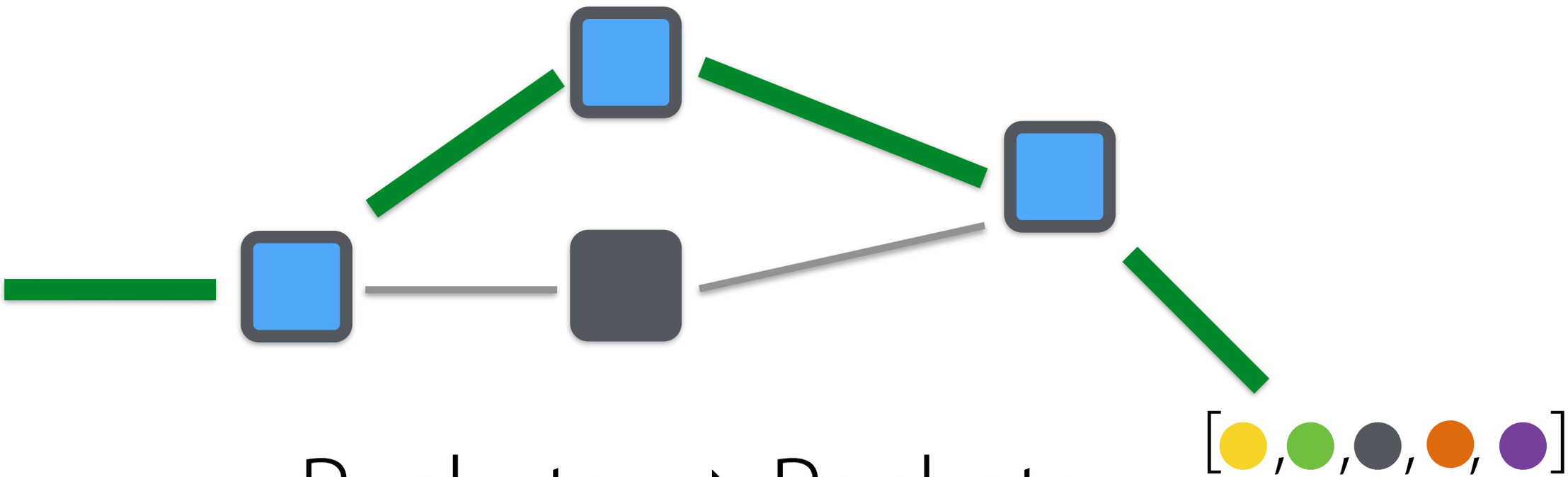

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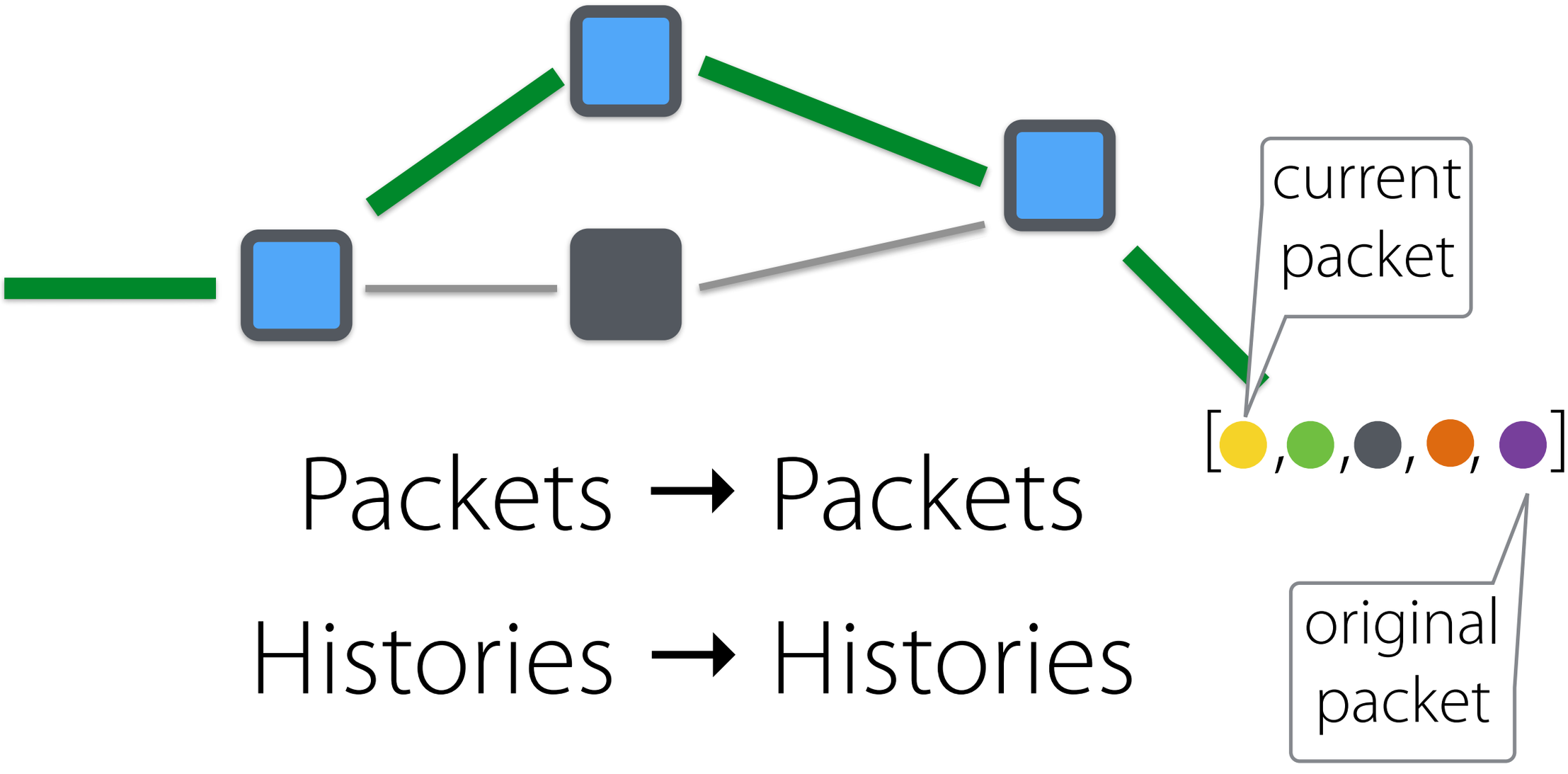

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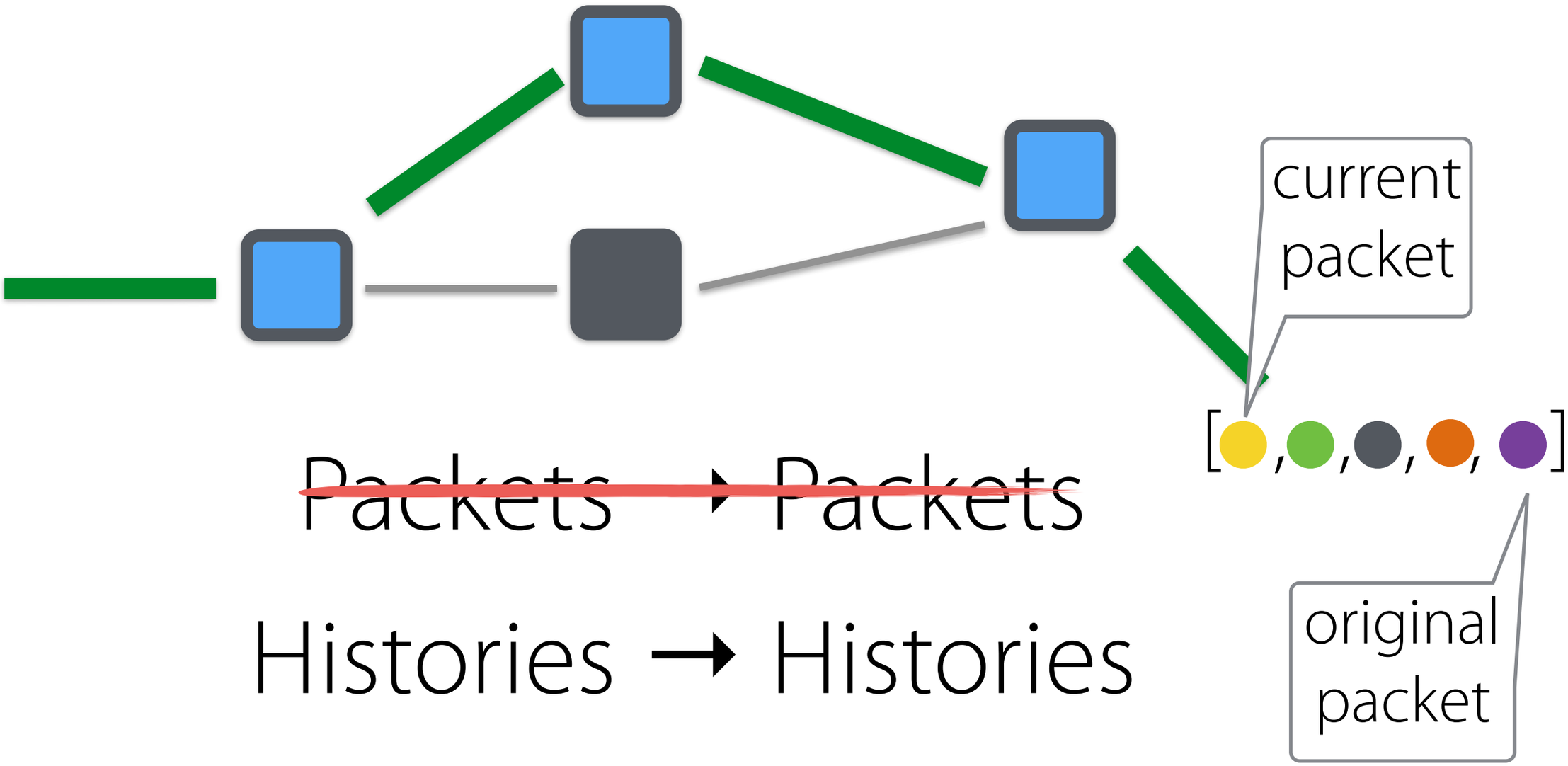

~~Packets → Packets~~

Histories → Histories

NetKAT Syntax

pol ::= **false**
| **true**
| f **=** n
| f **:=** n
| pol_1 **+** pol_2
| pol_1 • pol_2
| !pol
| pol*
| **dup**

NetKAT Syntax

pol ::= **false**
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Boolean Predicates

+

Regular Expressions

+

Packet Primitives

Negation may only be applied to B*oolean predicates:*
true, **false**, f **=** n, closed under +, •, and **!**

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Boolean Predicates + Regular Expressions + Packet Primitives } KAT

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Boolean Predicates + Regular Expressions + Packet Primitives

KAT

NetKAT

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NetKAT Syntax

pol ::= **false**
| **true**
| f **=** n
| pol **+** pol
| pol **•** pol
| **!**pol
| **dup**

Boolean Predicates

Primitives

if b **then** p_1 **else** p_2 ≜ (b • p_1) **+** (**!**b • p_2)

while b **do** p ≜ (b • p)* • **!**b

S➙S′ ≜ sw**=**S • **dup** • sw**:=**S′ • **dup**

Negation may only be applied to B*oolean predicates:*
true, **false**, f **=** n, closed under +, •, and **!**

Related: Kat + B! [LICS '14]

p ::= **false**
| **true**
| **t**?
| $\overline{\mathbf{t}}$?
| **t**!
| $\overline{\mathbf{t}}$!
| p_1 + p_2
| $p_1 \bullet p_2$
| !p
| p^*

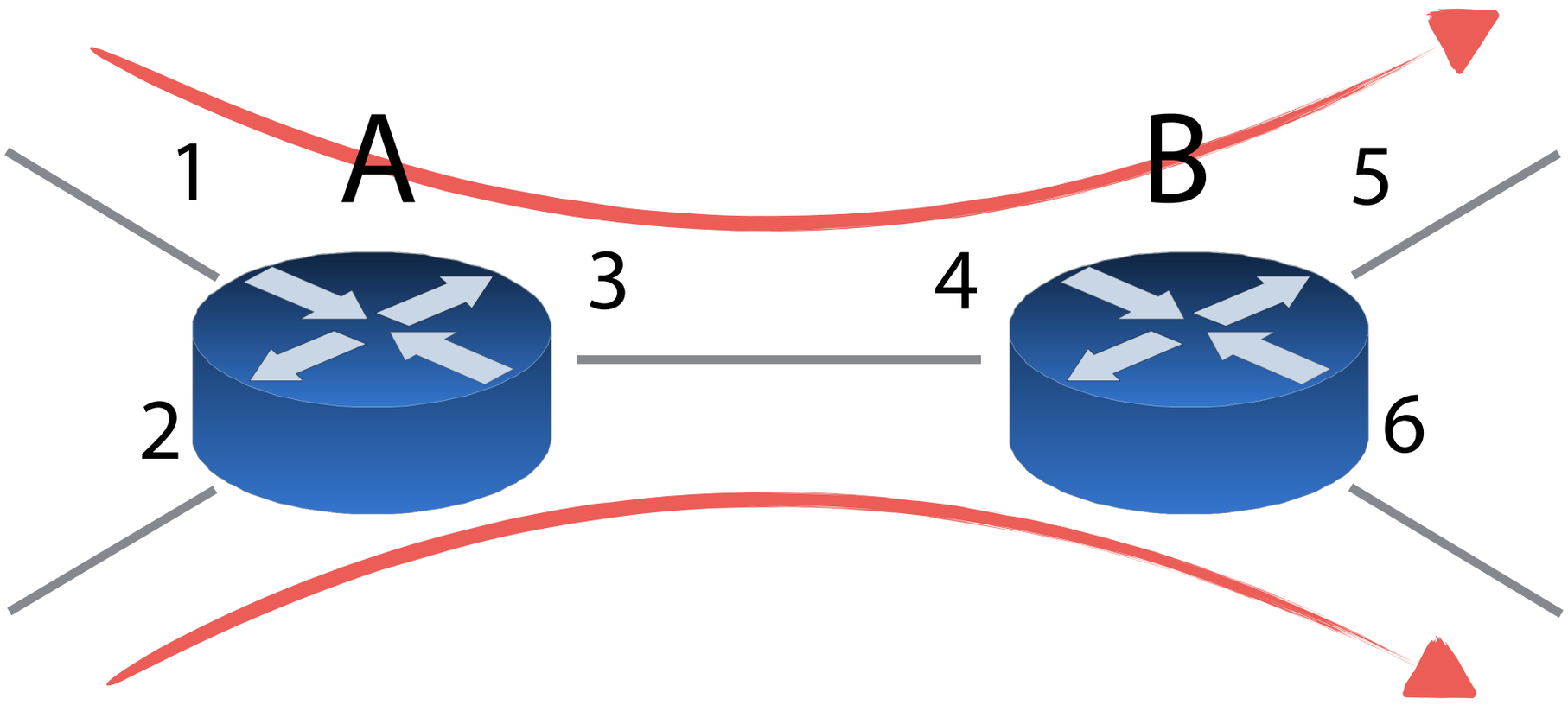
Idea: extend KAT with a finite number of mutable boolean state variables

Formally:

- Fix a finite set of tests $T = \{ t_1, \ldots t_n \}$
- Add primitive tests, both positive (**t**?) and negative ($\overline{\mathbf{t}}$?)
- Add primitive actions, both positive (**t!**) and negative ($\overline{\mathbf{t}}$!)

15-minute Break

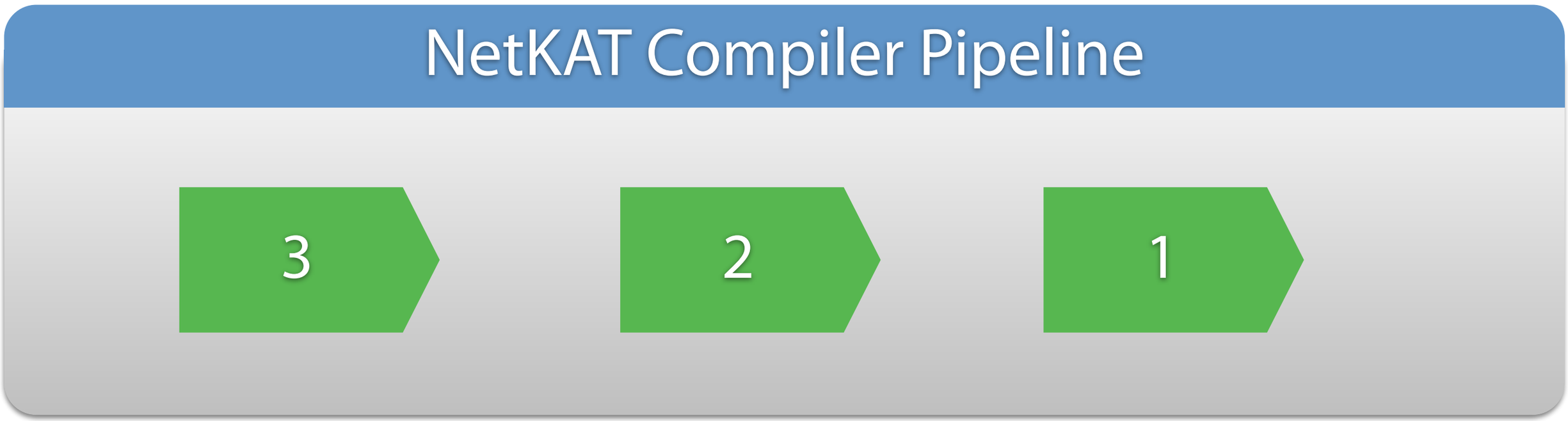
Question: How can we compile global programs like this?



port=1**; A→B;** port**:=**5
+
port=2**; A→B;** port**:=**6

Compilation

Overview



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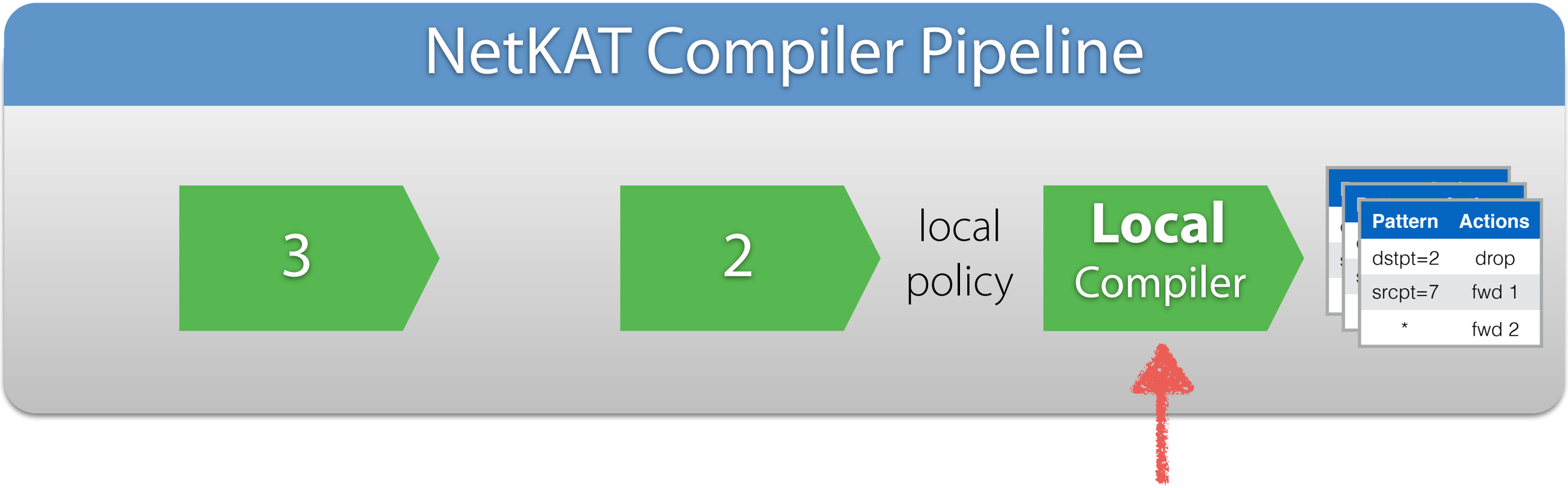
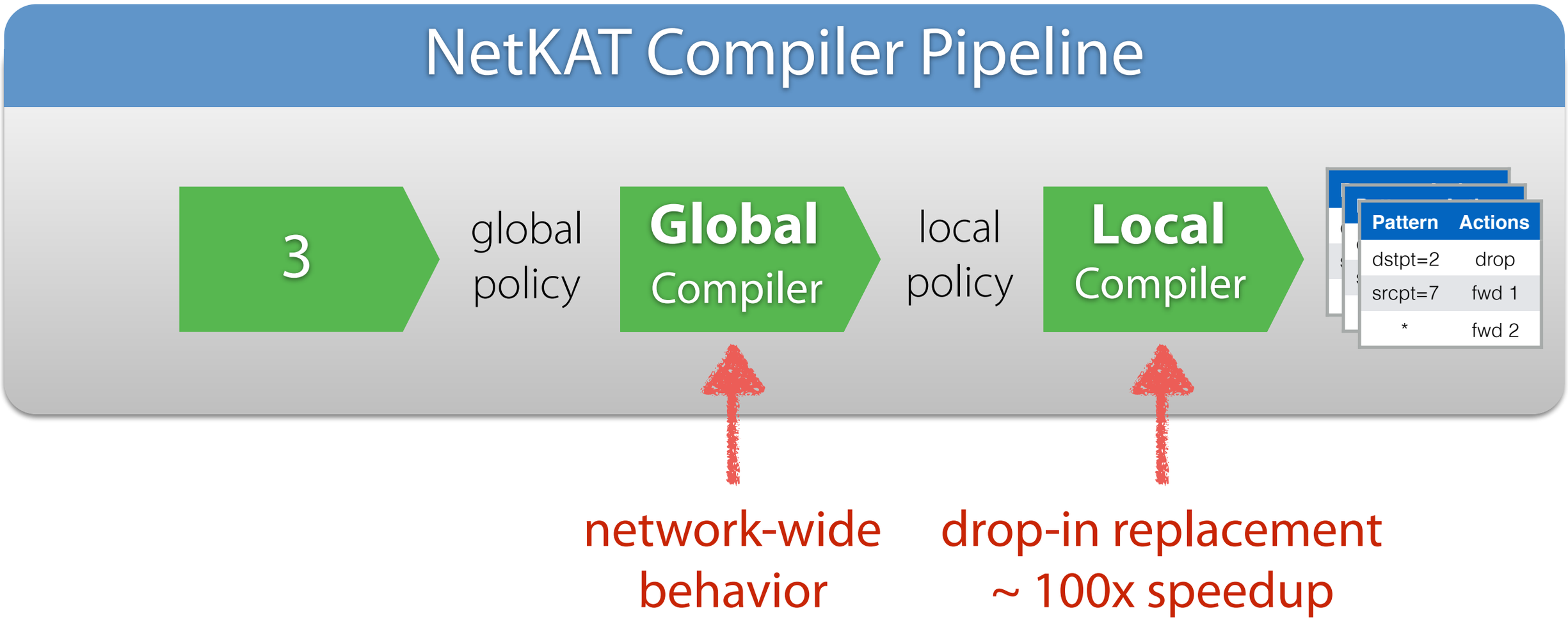
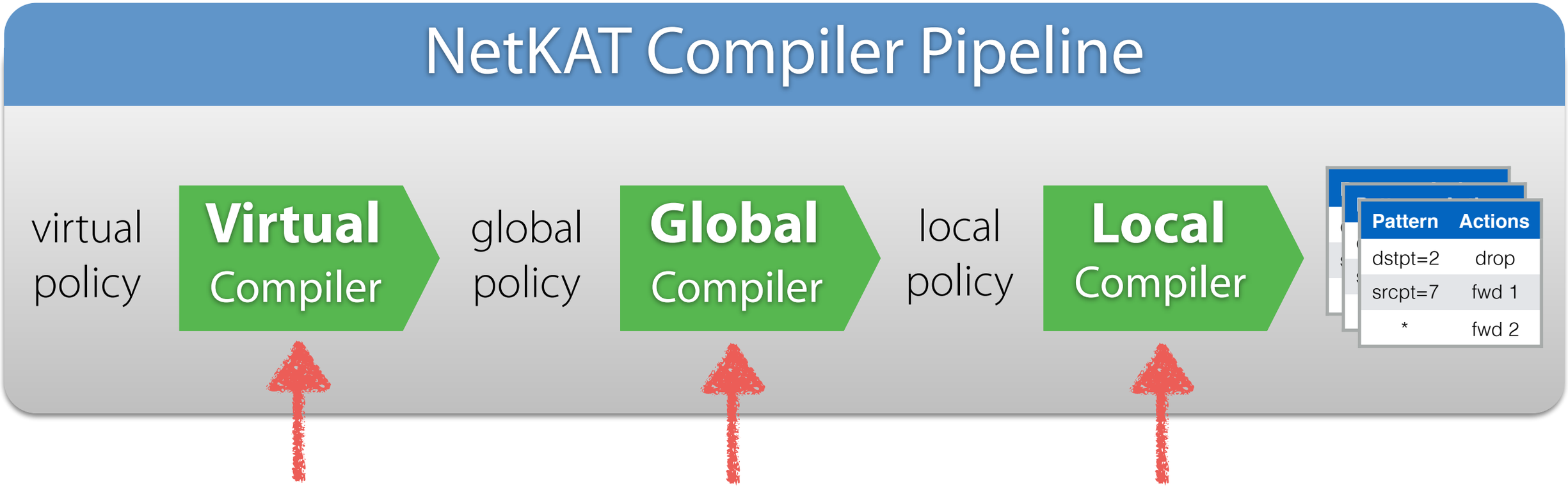


drop-in replacement
~ 100x speedup

Overview


NetKAT Compiler Pipeline
3
global policy
Global Compiler
local policy
Local Compiler
Pattern
Actions
dstpt=2
drop
srcpt=7
fwd 1
*
fwd 2
network-wide behavior
drop-in replacement
~ 100x speedup

Overview



abstract topologies

network-wide behavior

drop-in replacement
~ 100x speedup

Overview

NetKAT Compiler Pipeline

virtual policy

Virtual Compiler

global policy

Global Compiler

local policy

Local Compiler

Pattern	Actions
dstpt=2	drop
srcpt=7	fwd 1
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abstract topologies

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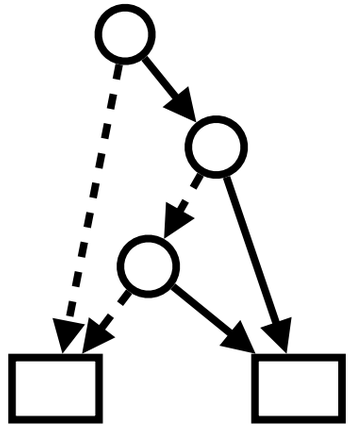
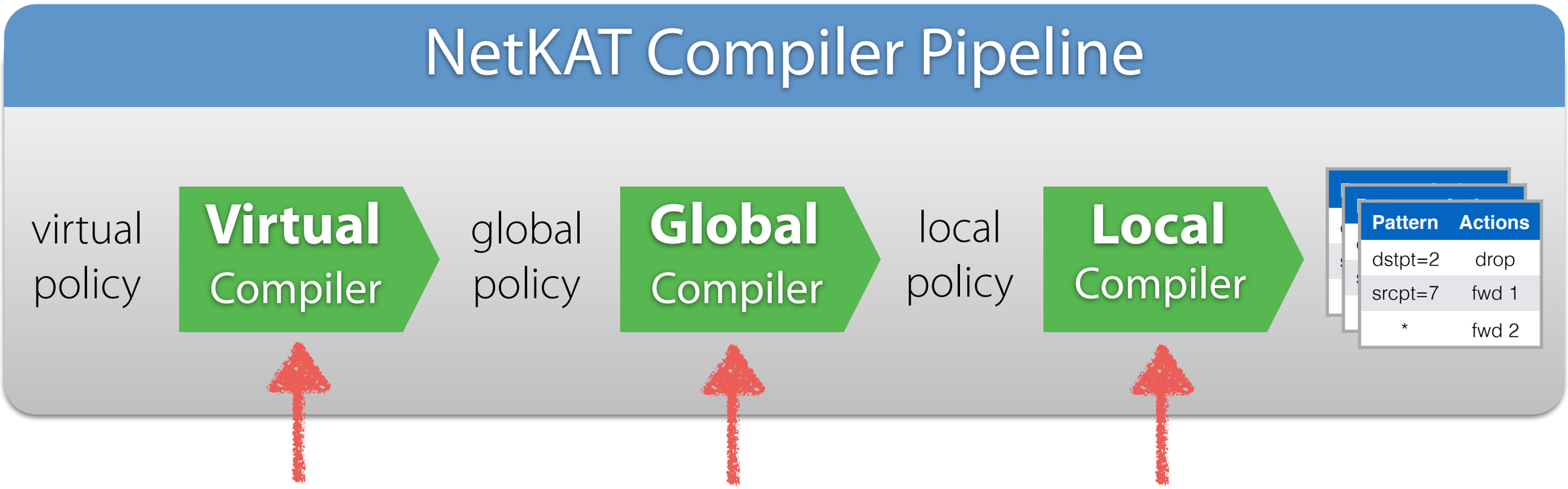


Overview



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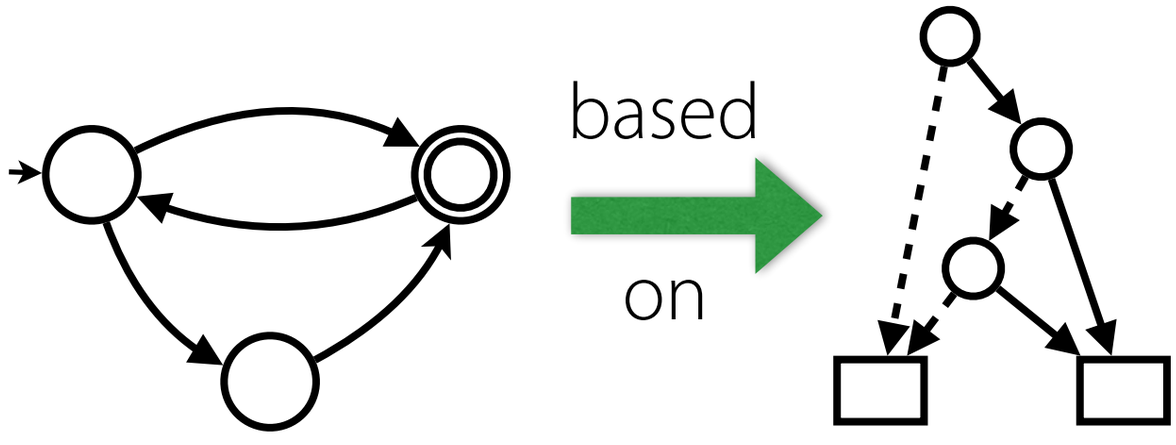

Overview

NetKAT Compiler Pipeline

virtual policy

Virtual Compiler

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dstpt=2	drop
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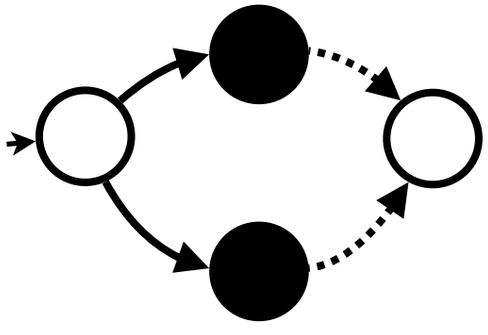
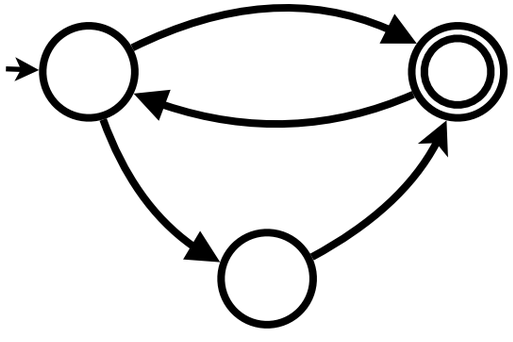
abstract topologies

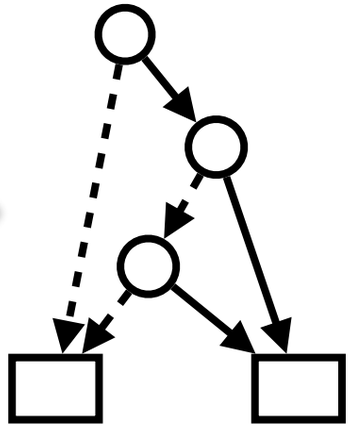


network-wide behavior



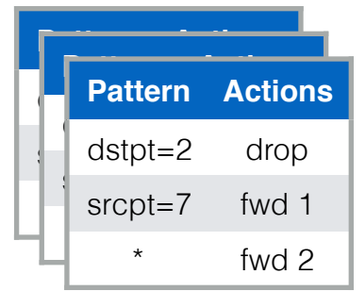
based on

drop-in replacement ~ 100x speedup



Local Compilation

Virtual Compiler → **Global** Compiler → **Local** Compiler



Input: local program

Output: collection of flow tables, one per switch

Challenges: efficiency and size of generated tables

Local Compilation

Input: local program

Output: collection of flow tables, one per switch

Challenges: efficiency and size of generated tables

Strawman Approach

```
let route =
  if ipDst = 10.0.0.1 then
    port := 1
  else if ipDst = 10.0.0.2 then
    port := 2
  else
   port := learn
```

+

```
let monitor =
if (tcpSrc = 22 + tcpDst = 22) then
  port:=console
else
  false
```

Strawman Approach

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let route =
  if ipDst = 10.0.0.1 then
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Pattern	Actions
src=10.0.0.1	Fwd 1
src=10.0.0.2	Fwd 2
*	Controller

Pattern	Actions
tcpSrc=22	Controller
tcpDst=22	Controller
*	Drop

Strawman Approach

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let route =
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+

Pattern	Actions
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tcpDst=22	Controller
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Strawman Approach

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Pattern	Actions
src=10.0.0.1	Fwd 1
src=10.0.0.2	Fwd 2
*	Controller

Pattern	Actions
tcpSrc=22	Controller
tcpDst=22	Controller
*	Drop

Inefficient!

Tables are a hardware abstraction,
not an efficient data structure!!

Better Approach

```
let route =
  if ipDst = 10.0.0.1 then
    port := 1
  else if ipDst = 10.0.0.2 then
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  else
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```

+

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Better Approach

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let monitor =
if (tcpSrc = 22 + tcpDst = 22) then
  port:=console
else
  false
```

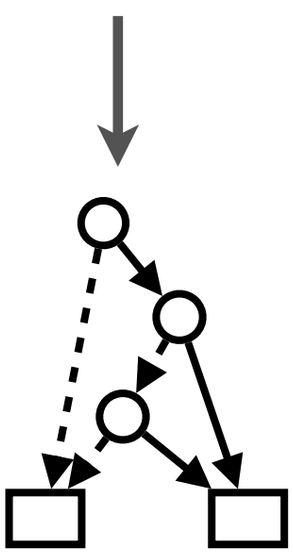

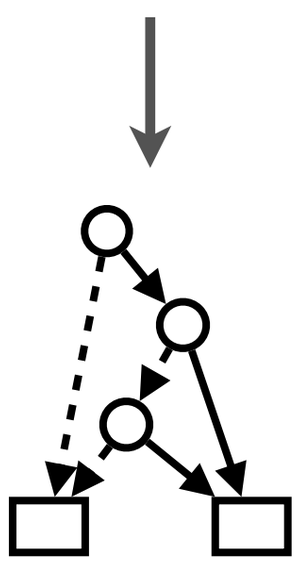

Better Approach

```
let route =
  if ipDst = 10.0.0.1 then
    port := 1
  else if ipDst = 10.0.0.2 then
    port := 2
  else
   port := learn
```

+

```
let monitor =
if (tcpSrc = 22 + tcpDst = 22) then
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```

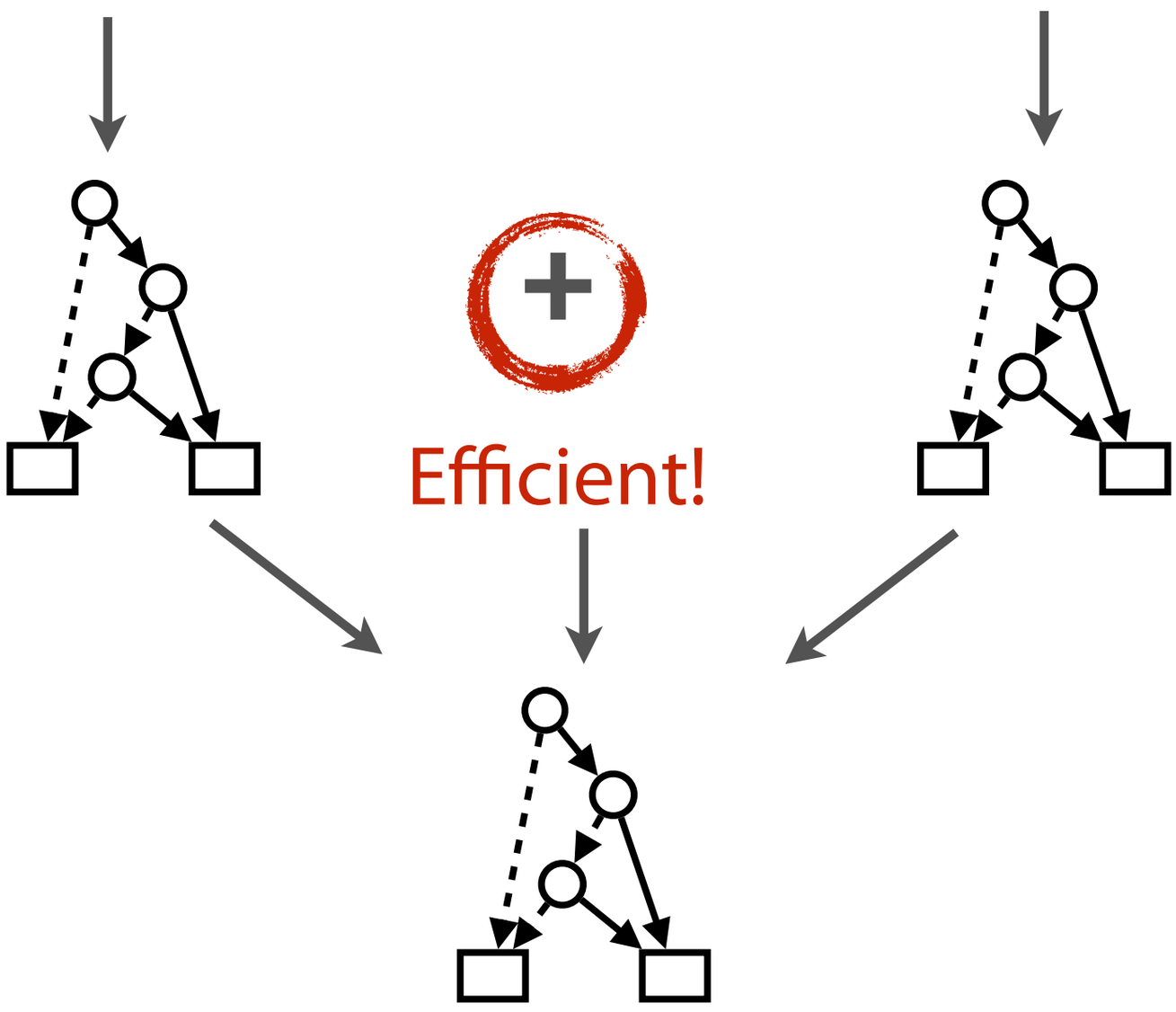

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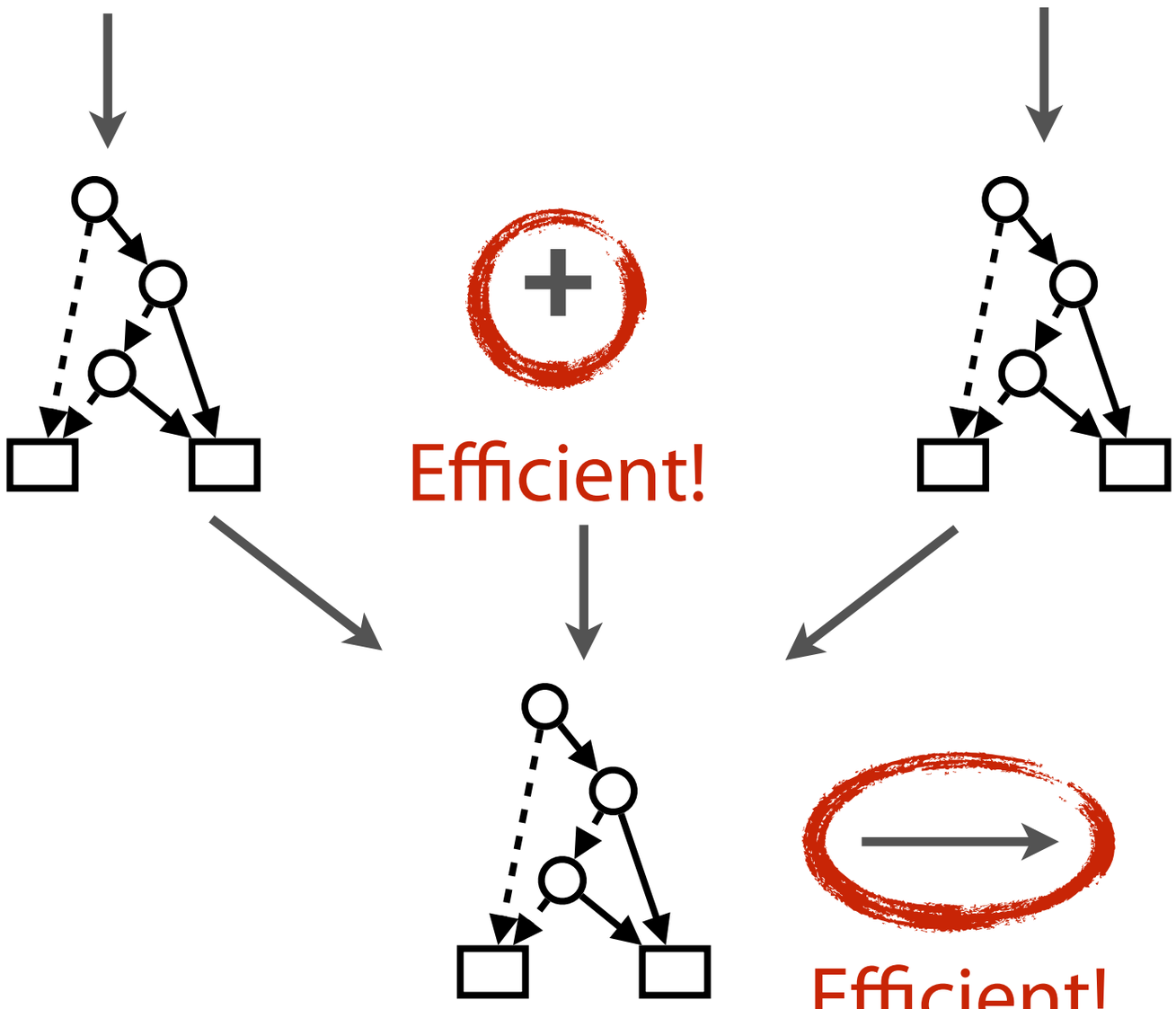
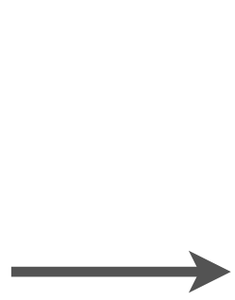
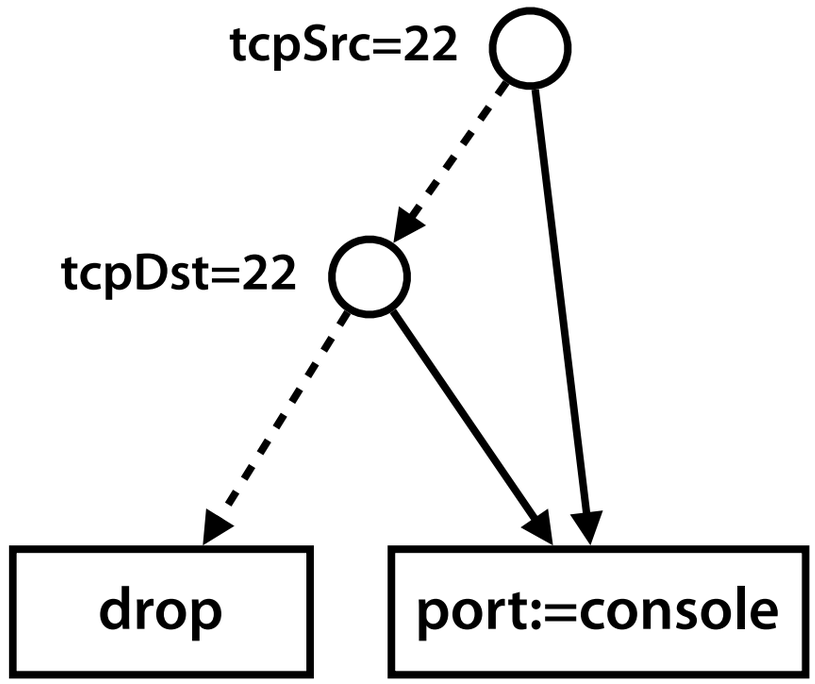


Pattern	Actions
ipDst=10.0.0.1, tcpSrc=22	Forward 1, Controller
ipDst=10.0.0.1, tcpDst=22	Forward 1, Controller
…	

IR: Forwarding Decision Diagrams

```
if (tcpSrc = 22
  + tcpDst = 22)
then
  port := console
else
  drop
```





Inspired by Binary Decision Diagrams

IR: Forwarding Decision Diagrams

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then
  port := console
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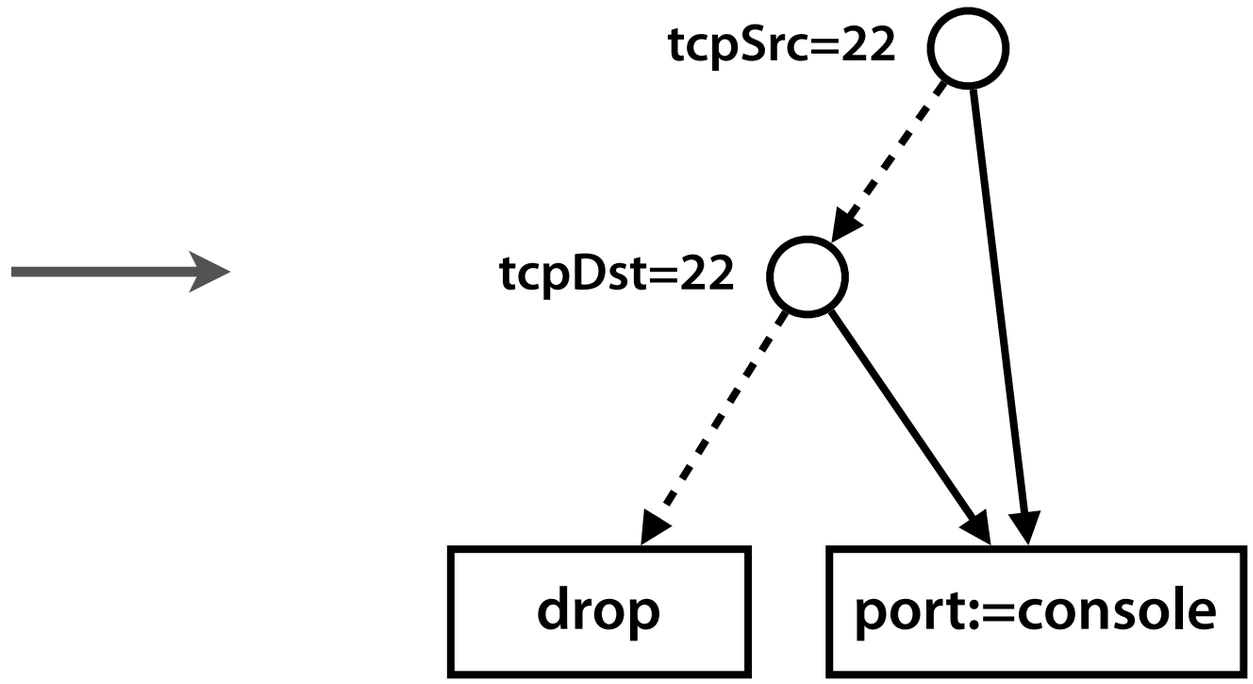
```



Inspired by Binary Decision Diagrams

NetKAT operators (**+**, **;** , *****, **!**) can be implemented efficiently on FDDs using standard BDD techniques

Global Compilation

Input: NetKAT program (with links)

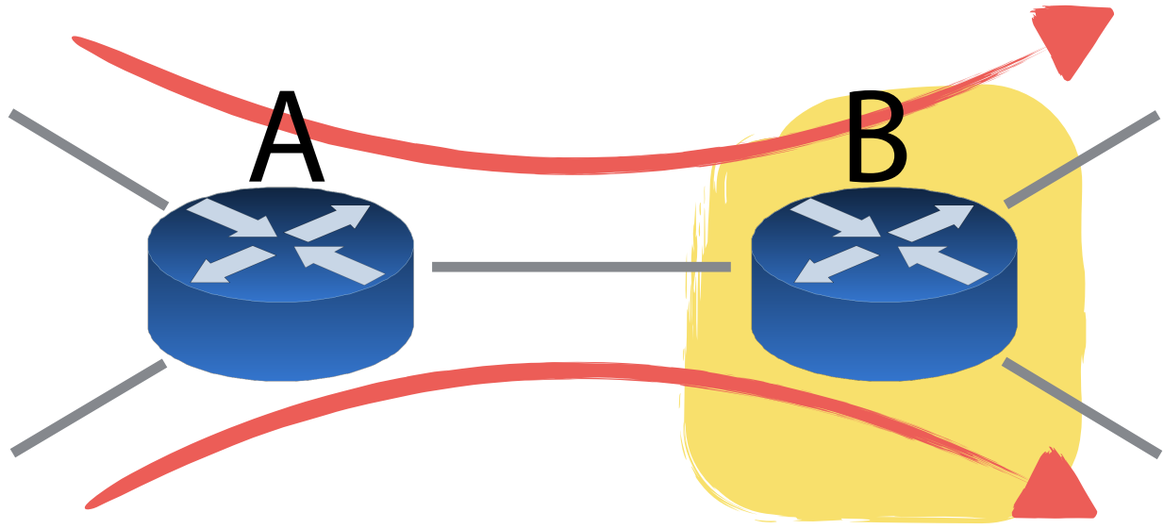
Output: equivalent local program (without links)

Global Compilation

Input: NetKAT program (with links)

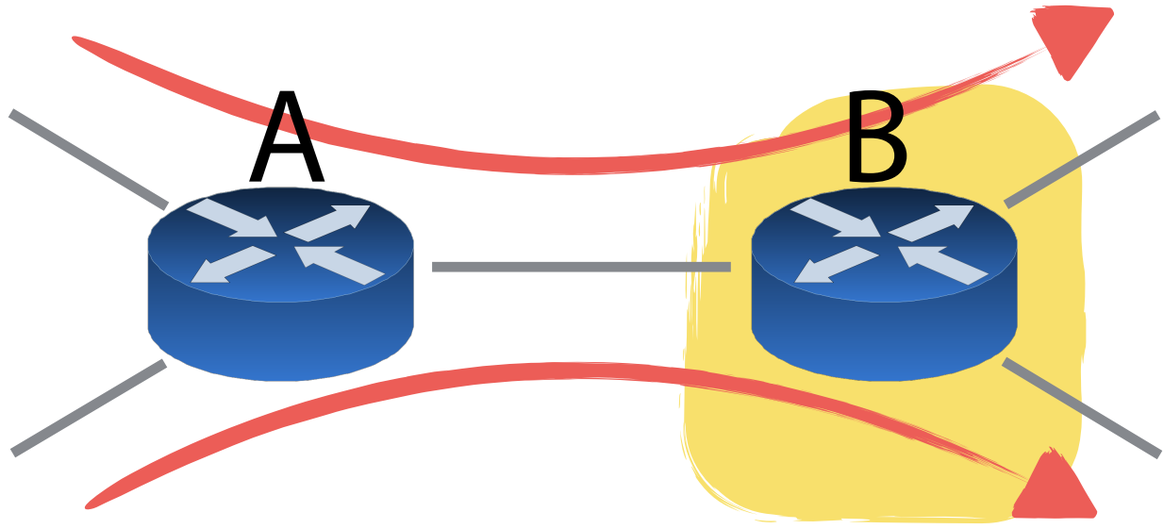
Output: equivalent local program (without links)

Main Challenges



Main Challenges

1. Adding Extra State
"Tagging"



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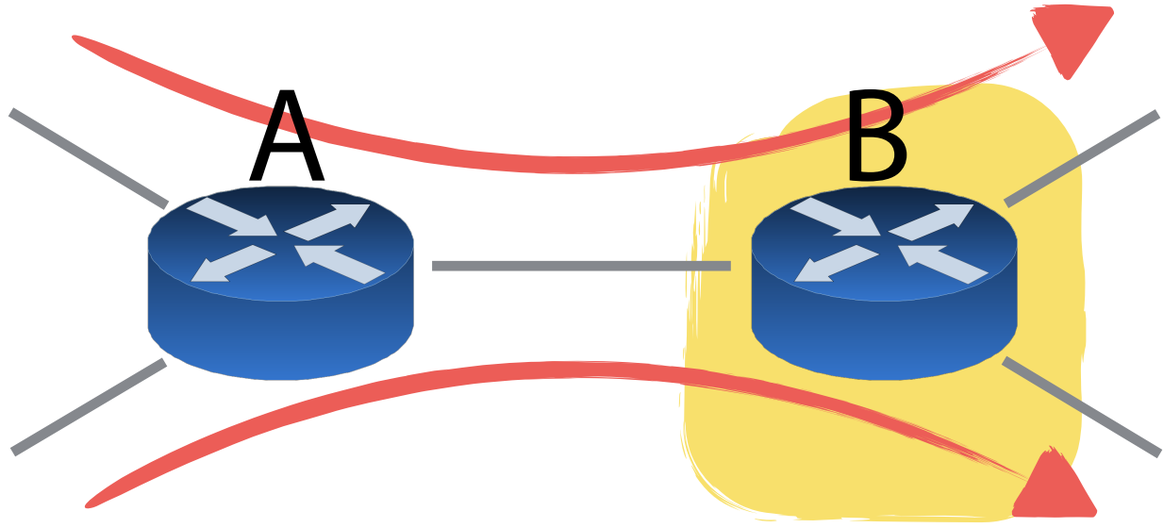
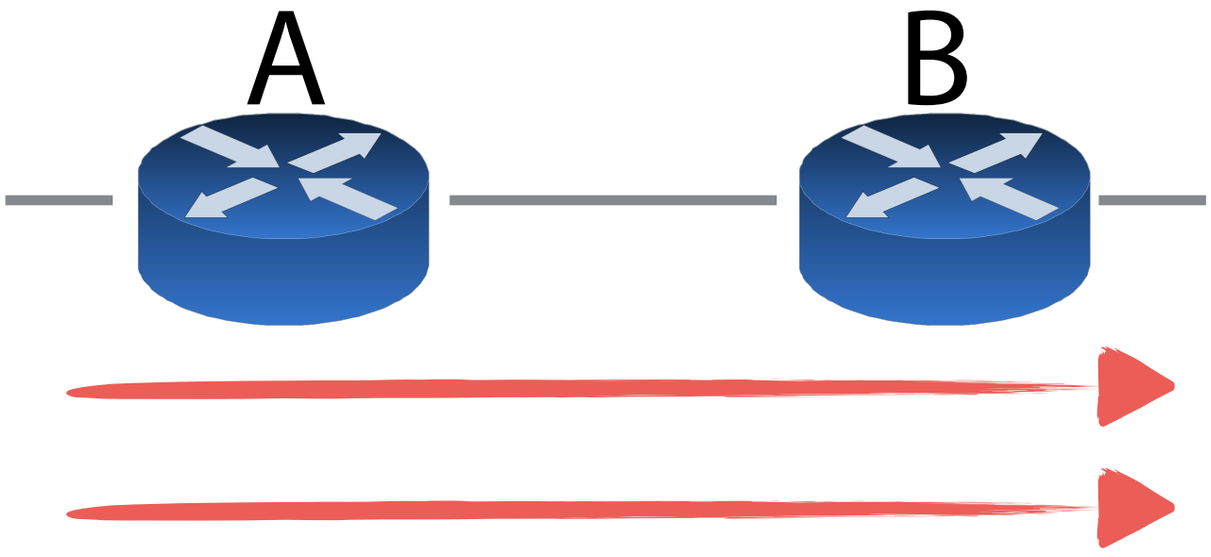
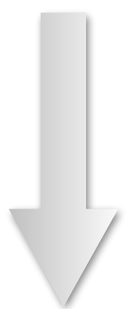


2. Avoiding Duplication
(naive tagging is unsound!)



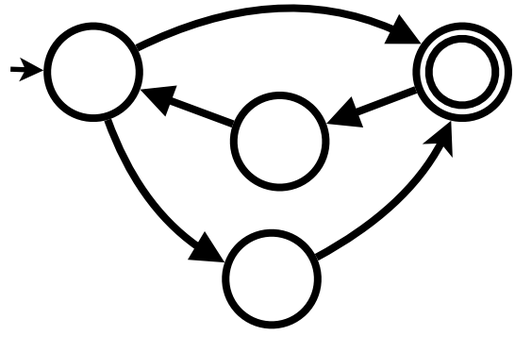
Our Solution

Our Solution



Adding Extra State
= Translation to Automaton

NetKAT NFA

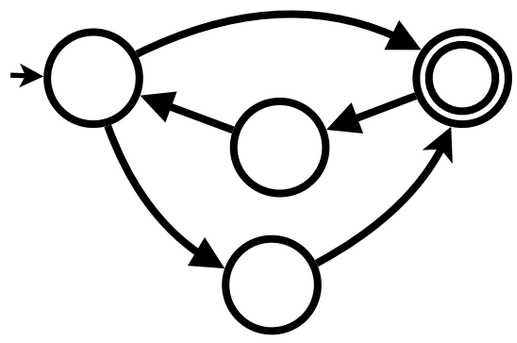


Our Solution

Global Program

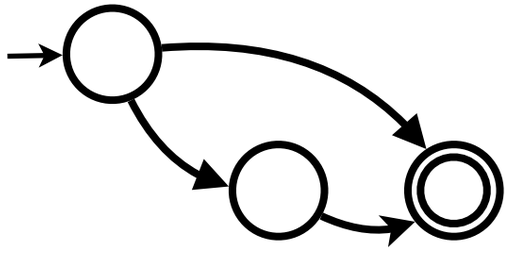
Adding Extra State
= Translation to Automaton

NetKAT NFA



Avoiding Duplication
= Determinization

NetKAT DFA



Our Solution

Global Program

Adding Extra State
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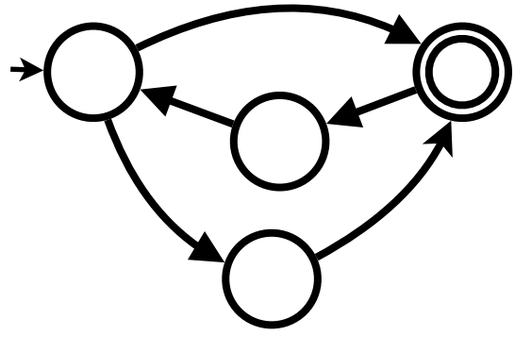

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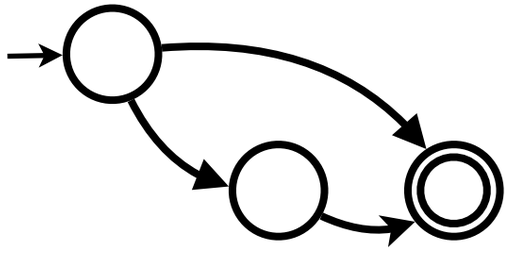

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Global Program

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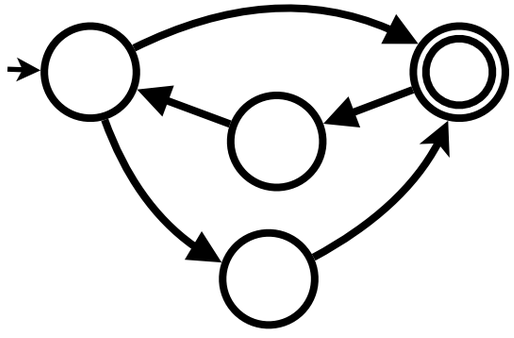
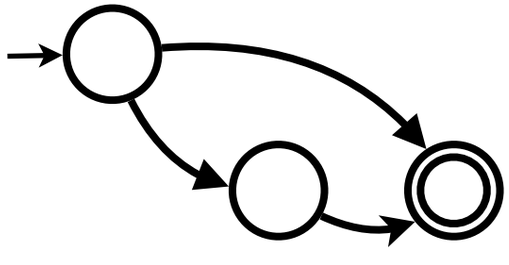


Automaton Minimization
= Tag Elimination

Avoiding Duplication
= Determinization

NetKAT DFA



Local Program

Virtual Compilation

Input: program written against virtual topology

Output: global program that simulates virtual behavior

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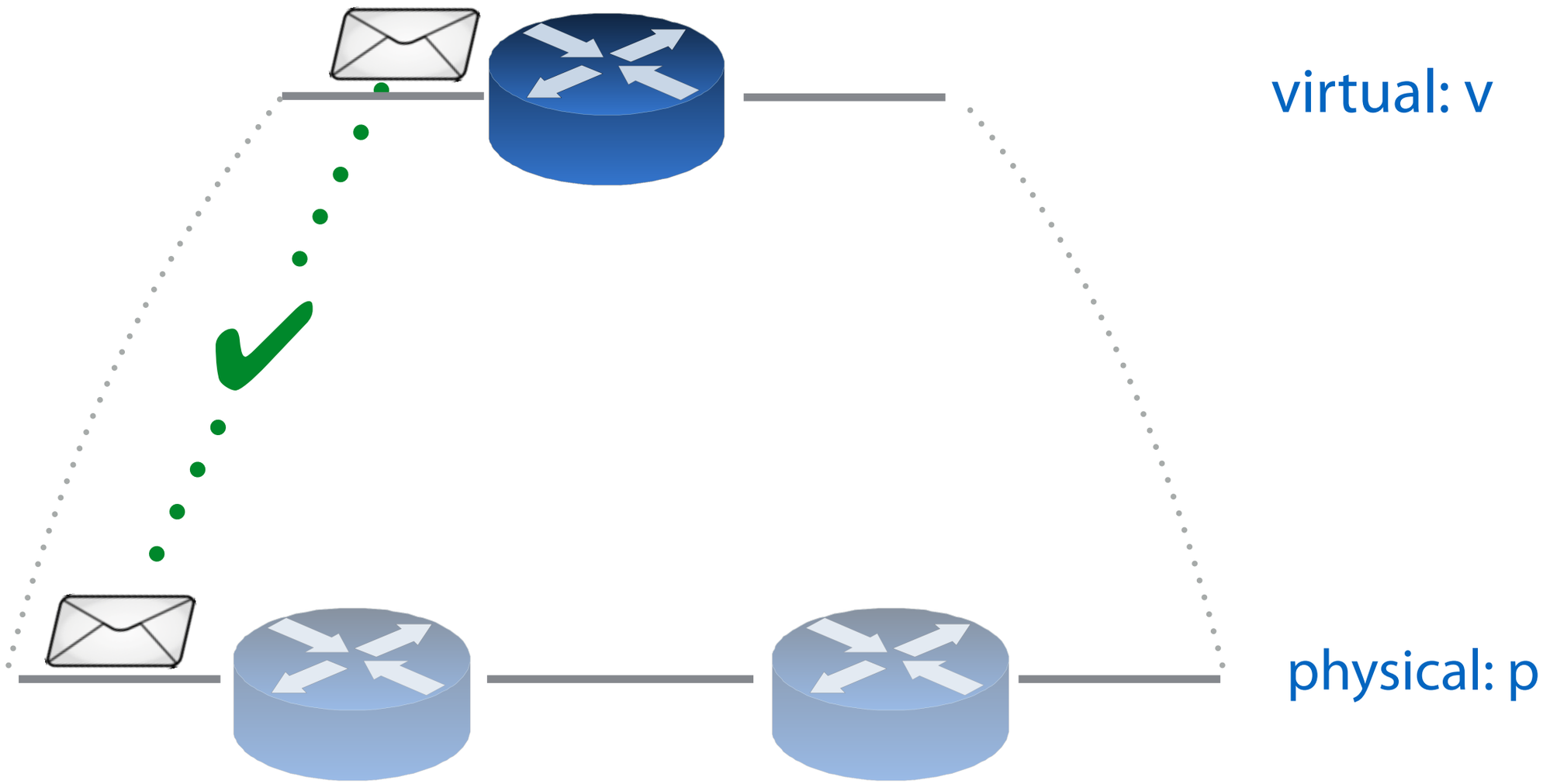
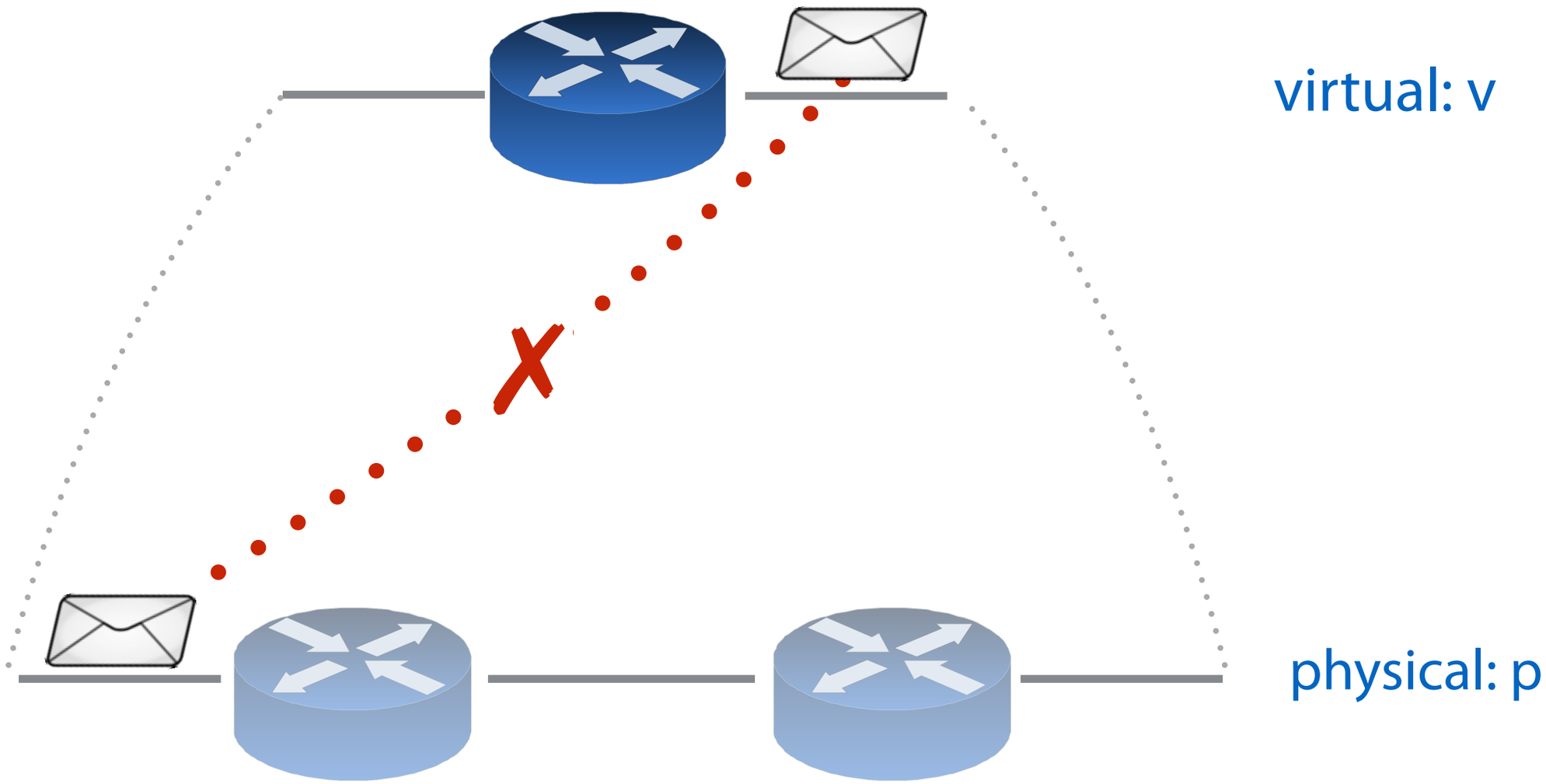
Virtualization



virtual: v

physical: p

Virtualization



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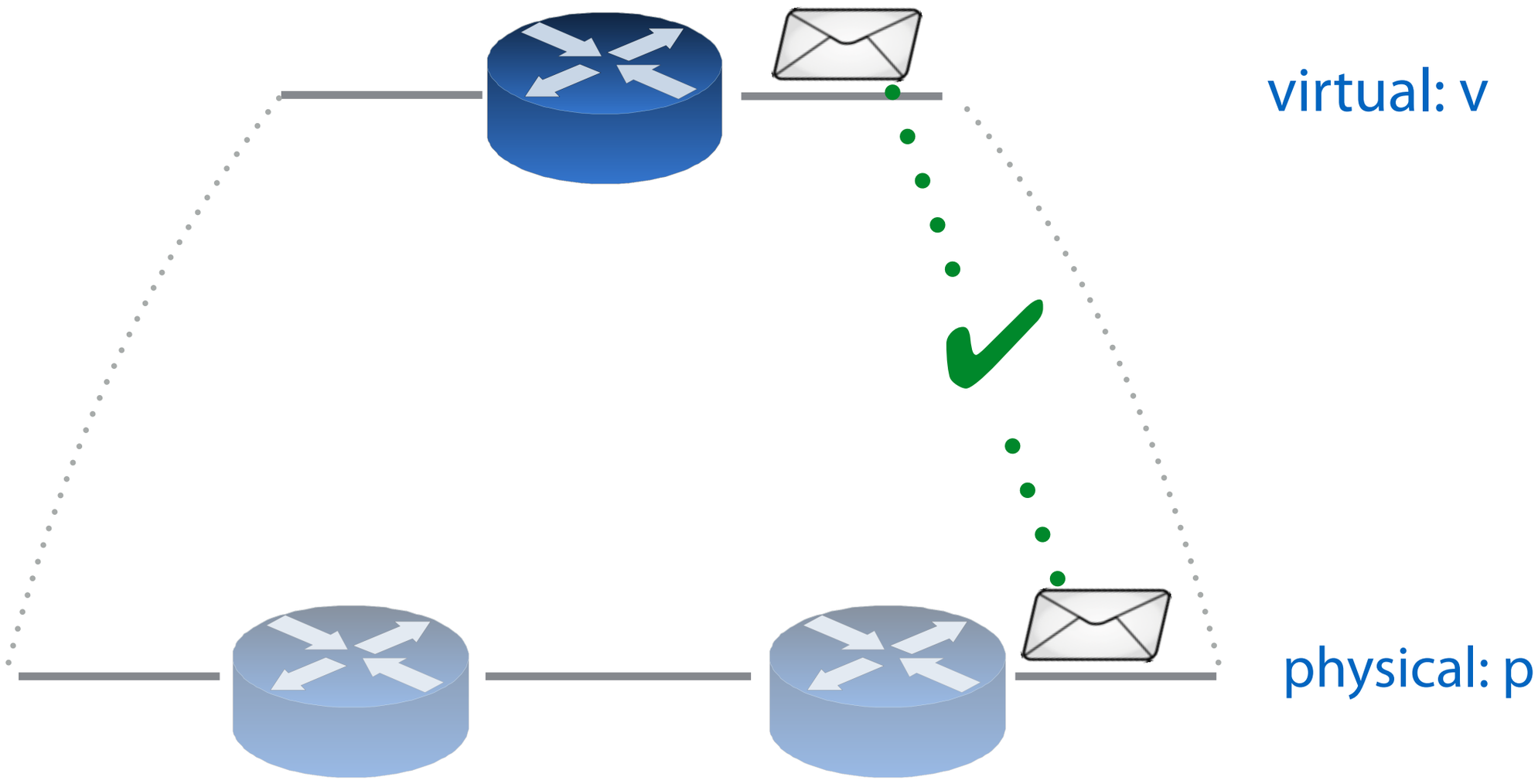
Virtualization



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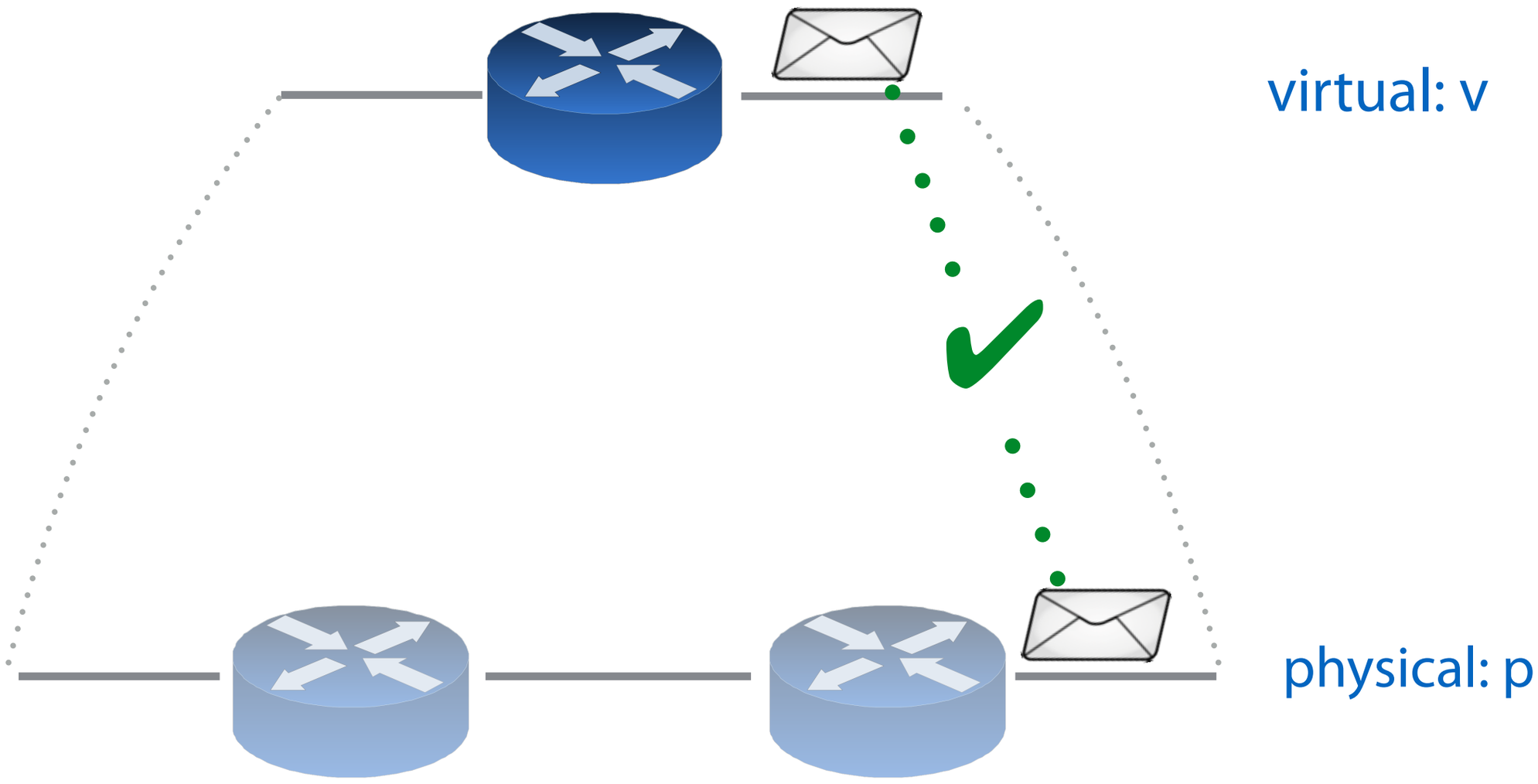
Virtualization



Observation: can formulate execution of a virtual program as a two-player game

Compiler: synthesizes physical program p that encodes a winning strategy to all instances of that game

NetKAT Semantics

Denotational

NetKAT Semantics

Denotational

Axiomatic

$\vdash p \equiv q$

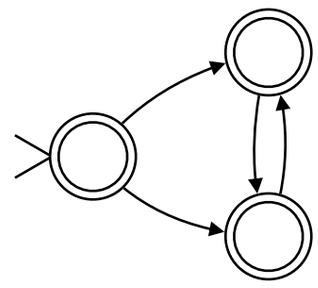
NetKAT Semantics

Denotational

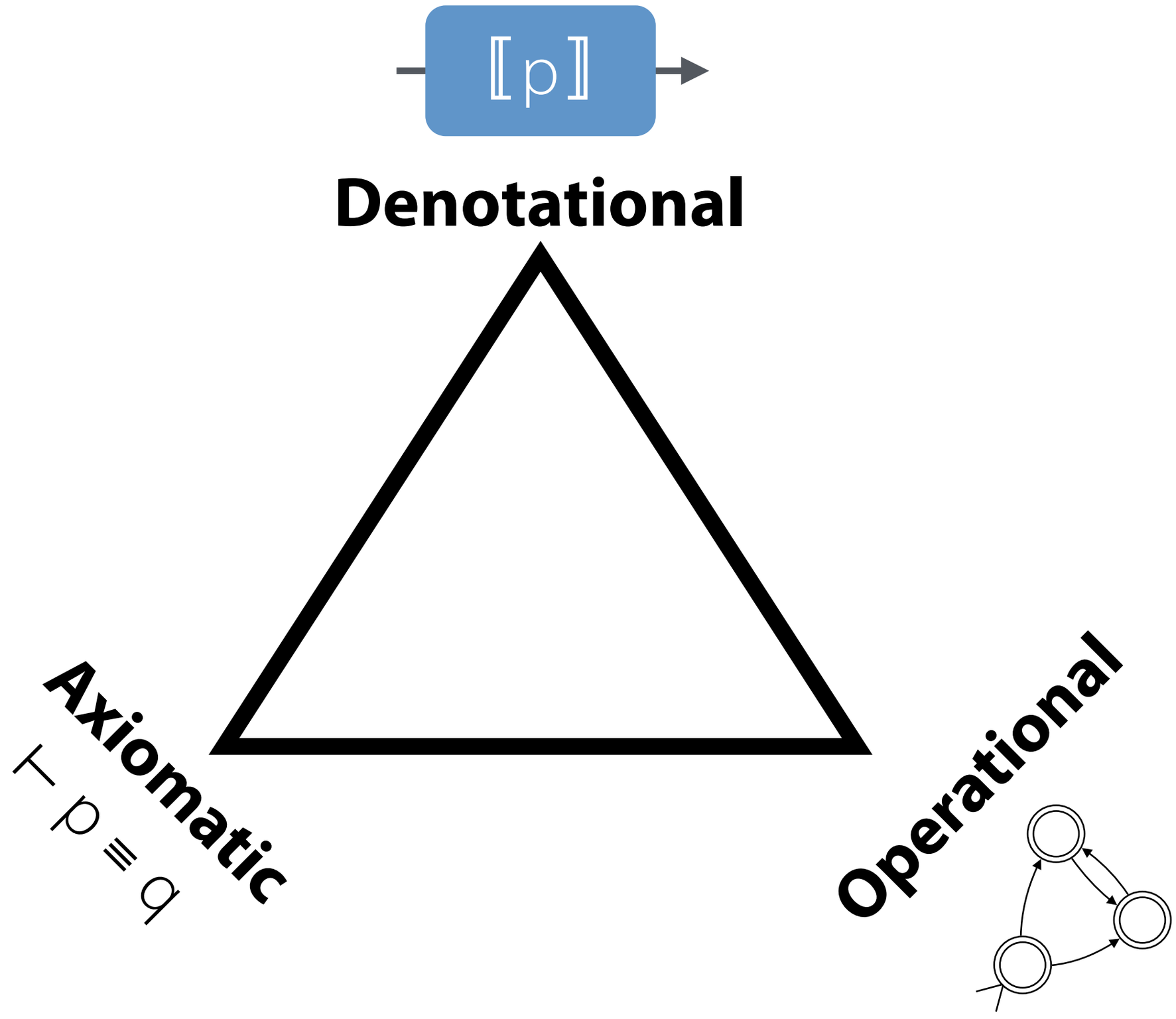
Axiomatic

$\vdash p \equiv q$

Operational



NetKAT Semantics



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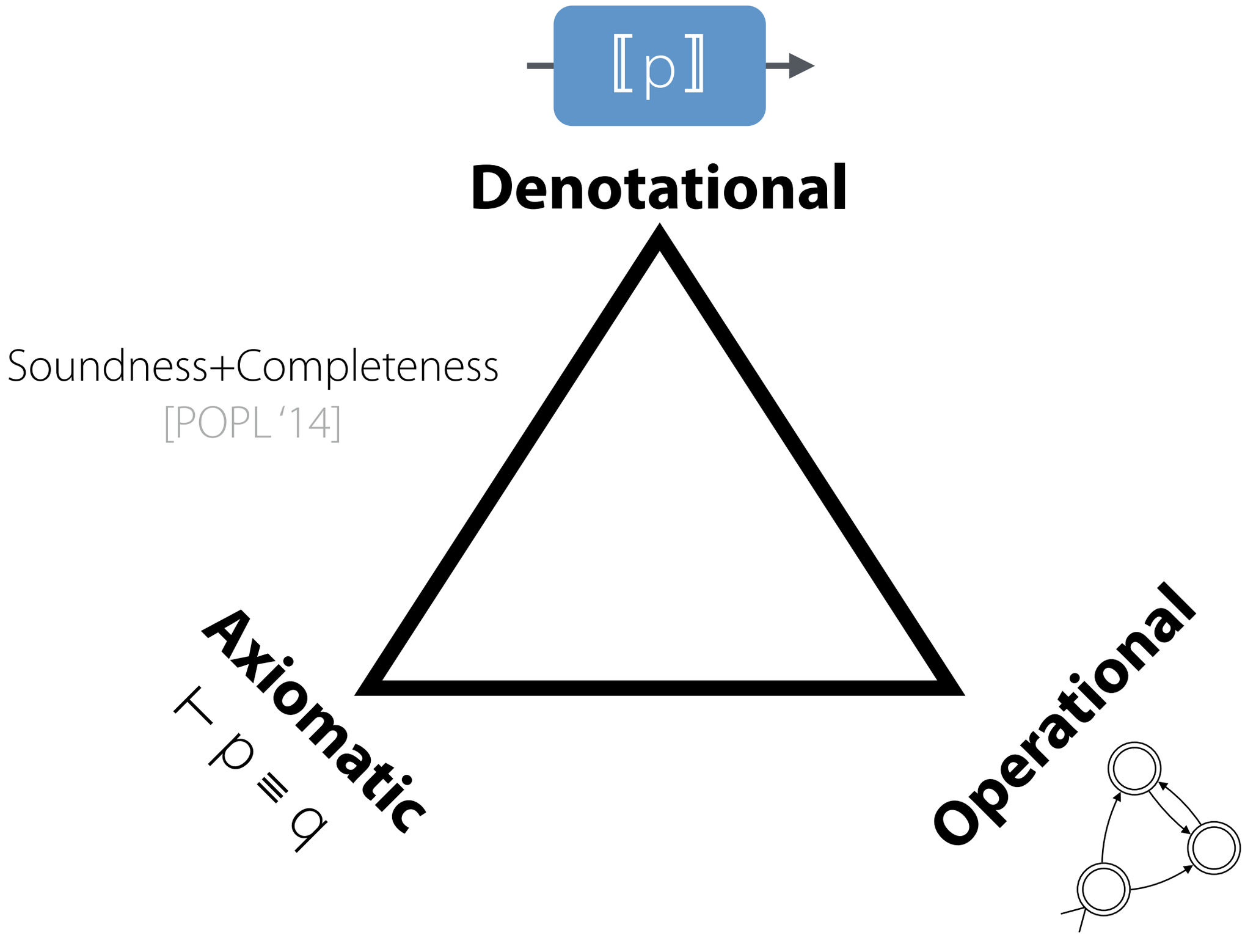

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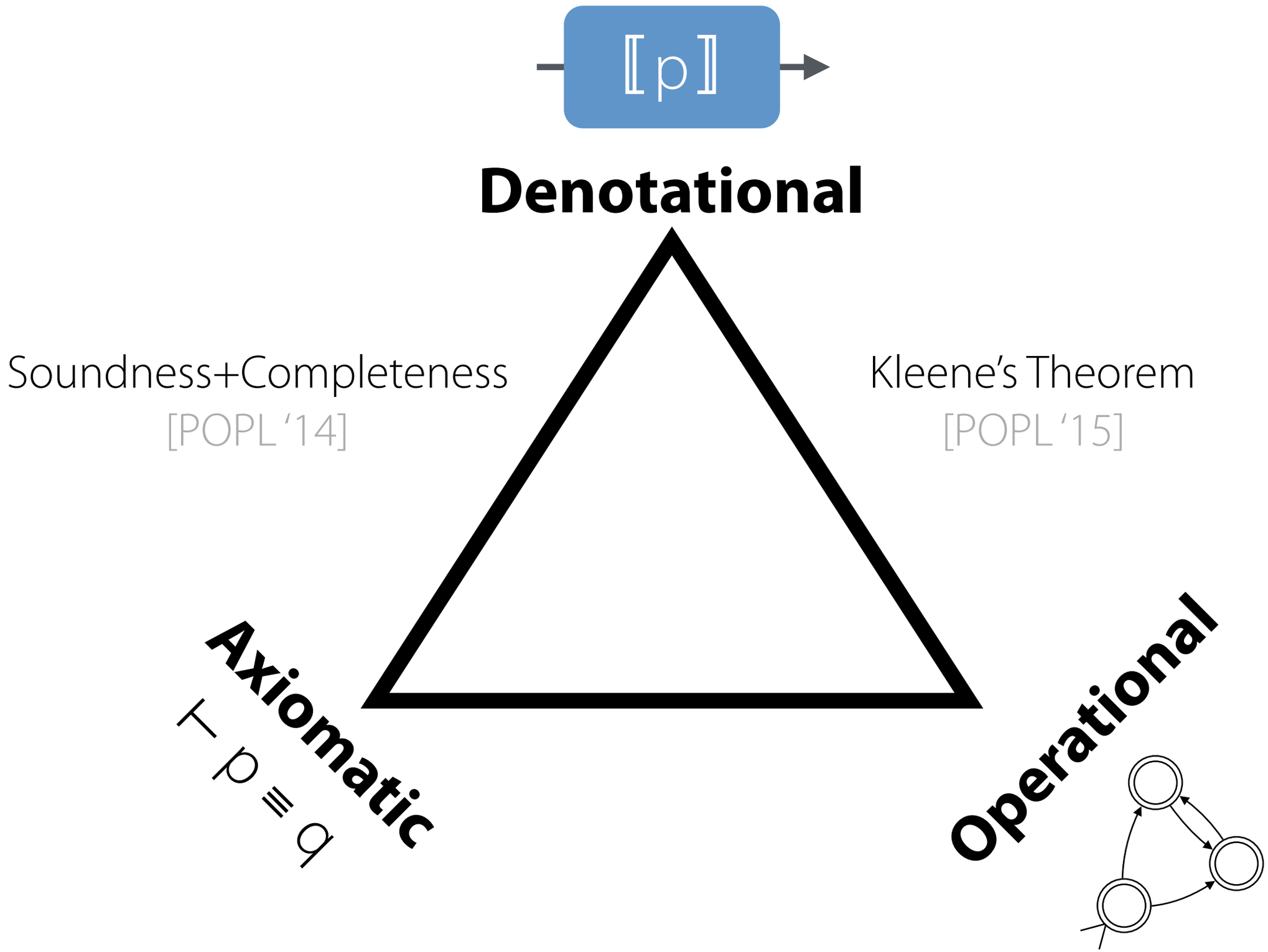
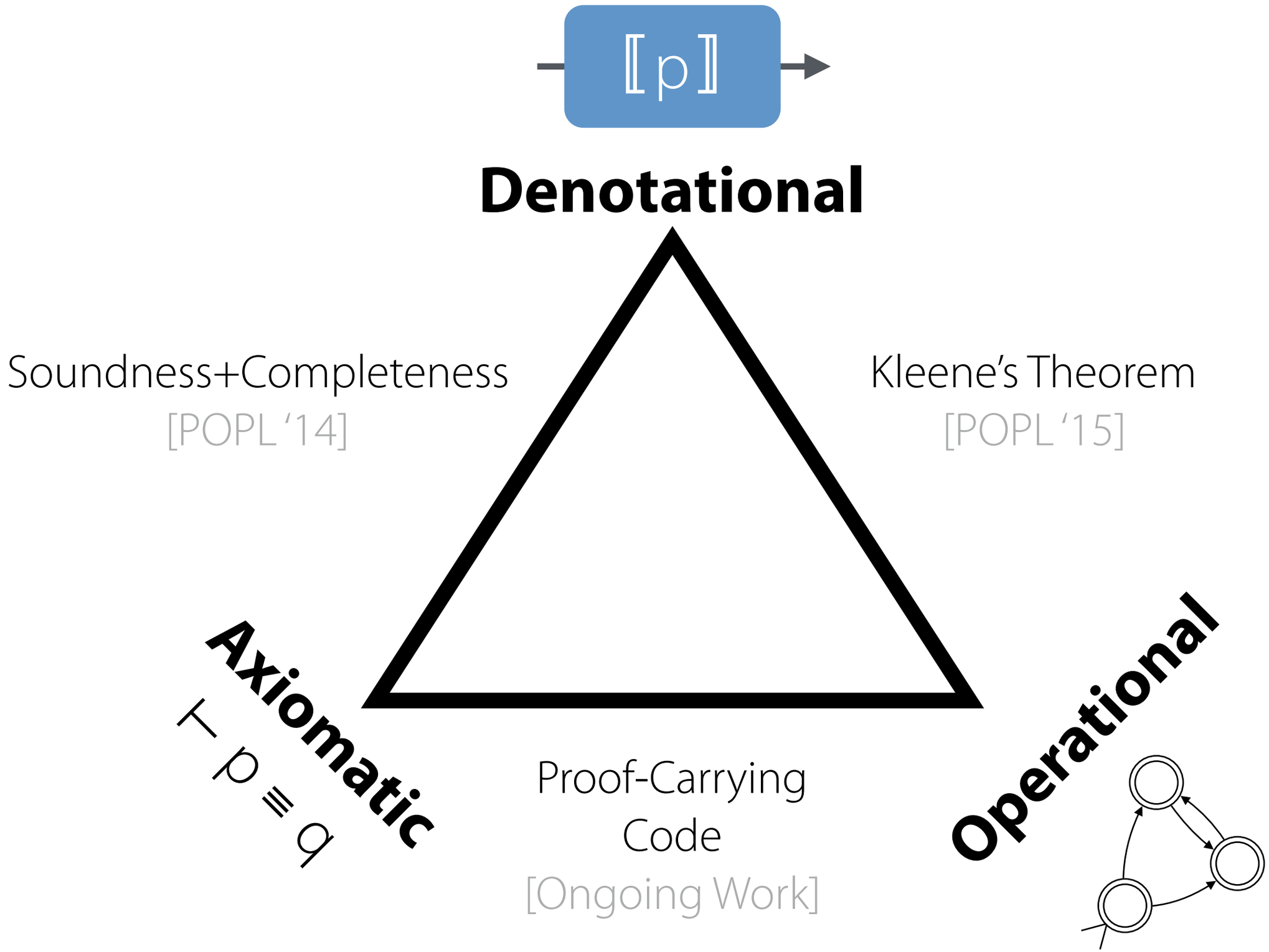


NetKAT Semantics



Denotational Semantics

Denotational Semantics

pol ::=
- | **false**
- | **true**
- | field **=** val
- | field **:=** val
- | pol_1 **+** pol_2
- | pol_1 • pol_2
- | !pol
- | pol*
- | **dup**

Denotational Semantics

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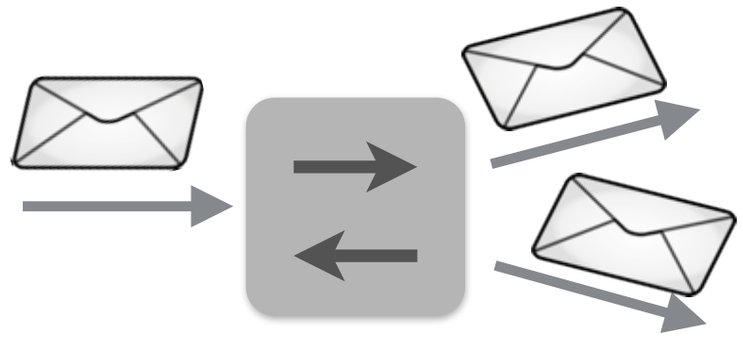
Local: input-output behavior of switches



⟦pol⟧ ∈ Packets → Packets

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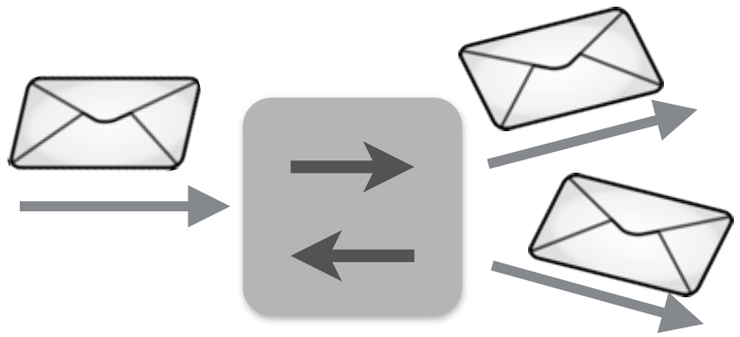
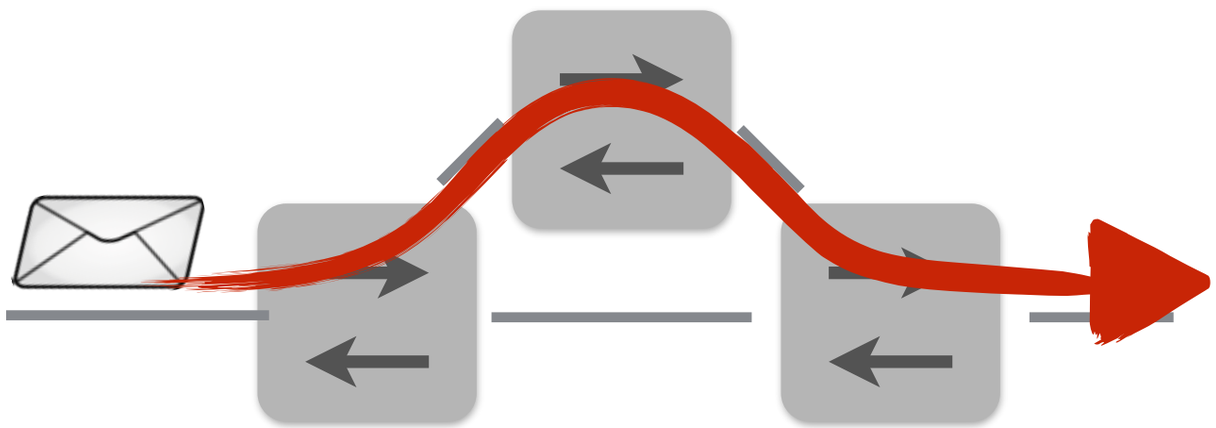


⟦pol⟧ ∈ Packets → Packets

Global: network-wide paths



⟦pol⟧ ∈ Histories → Histories

NetKAT Semantics

$[\![pol]\!] \in \textbf{History} \rightarrow \textbf{History Set}$

$[\![\textbf{true}]\!]\ h = \{ h \}$

$[\![\textbf{false}]\!]\ h = \{\}$

$$[\![f = n]\!]\ pk :: h = \begin{cases} \{ pk :: h \} & \text{if } pk.f = n \\ \{\} & \text{otherwise} \end{cases}$$

$[\![\textbf{!}\ pol]\!]\ h = \{ h \} \setminus [\![pol]\!]$

$[\![f := n]\!]\ pk :: h = \{ pk[f:=n] :: h \}$

$[\![pol_1 + pol_2]\!]\ h = [\![pol_1]\!]\ h \cup [\![pol_2]\!]\ h$

$[\![pol_1 \bullet pol_2]\!]\ h = ([\![pol_1]\!] \bullet [\![pol_2]\!])\ h$

$[\![pol^*]\!]\ h = (\ \bigcup_i [\![pol]\!]^i\ h)$

$$[\![\ S \rightarrow S']\!]\ pk :: h = \begin{cases} \{ pk[sw:=S'] :: pk :: h \} & \text{if } pk.sw = S \\ \{\} & \text{otherwise} \end{cases}$$

$f,g \in \textbf{History} \rightarrow \textbf{History Set}$

$(f \bullet g)\ h = \bigcup \{ g\ h' \mid h' \in f\ h \}$

Axiomatic Semantics

Kleene Algebra with Tests

The design of NetKAT is not an accident!

Its foundation rests upon canonical structure:

- Regular operators (**+**, •, and ***) encode network paths
- Boolean operators (**+**, •, and **!**) encode switch tables

Need to extend KAT with additional axioms to obtain a sound and complete system

NetKAT Equational Axioms

Kleene Algebra Axioms

$p + (q + r) \equiv (p + q) + r$

$p + q \equiv q + p$

$p + \mathbf{false} \equiv p$

$p + p \equiv p$

$p \cdot (q \cdot r) \equiv (p \cdot q) \cdot r$

$p \cdot (q + r) \equiv p \cdot q + p \cdot r$

$(p + q) \cdot r \equiv p \cdot r + q \cdot r$

$\mathbf{true} \cdot p \equiv p$

$p \equiv p \cdot \mathbf{true}$

$\mathbf{false} \cdot p \equiv \mathbf{false}$

$p \cdot \mathbf{false} \equiv \mathbf{false}$

$\mathbf{true} + p \cdot p^* \equiv p^*$

$\mathbf{true} + p^* \cdot p \equiv p^*$

$p + q \cdot r + r \equiv r \Rightarrow p^* \cdot q + r \equiv r$

$p + q \cdot r + q \equiv q \Rightarrow p \cdot r^* + q \equiv q$

Boolean Algebra Axioms

$a + (b \cdot c) \equiv (a + b) \cdot (a + c)$

$a + \mathbf{true} \equiv \mathbf{true}$

$a + \mathbf{!}\, a \equiv \mathbf{true}$

$a \cdot b \equiv b \cdot a$

$a \cdot \mathbf{!}a \equiv \mathbf{false}$

$a \cdot a \equiv a$

Packet Axioms

$f := n \cdot f' := n' \equiv f' := n' \cdot f := n \quad \text{if } f \neq f'$

$f := n \cdot f' = n' \equiv f' = n' \cdot f := n \quad \text{if } f \neq f'$

$f := n \cdot f = n \equiv f := n$

$f = n \cdot f := n \equiv f = n$

$f := n \cdot f := n' \equiv f := n'$

$f = n \cdot f = n' \equiv \mathbf{false} \quad \text{if } n \neq n'$

$dup \cdot f = n \equiv f = n \cdot dup$

$\sum_i f = n_i \equiv \mathbf{true}$

NetKAT Equational Axioms

Kleene Algebra Axioms

$p + (q + r) \equiv (p + q) + r$

$p + q \equiv q + p$

$p + \mathbf{false} \equiv p$

$p + p \equiv p$

$p \bullet (q \bullet r) \equiv$

$p \bullet (q + r)$

$(p + q) \bullet$

$\mathbf{true} \bullet p \equiv$

$p \equiv p \bullet \mathbf{true}$

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$f = n \bullet f := n \equiv f = n$

$f := n \bullet f := n' \equiv f := n'$

$f = n \bullet f = n' \equiv \mathbf{false}$ if $n \neq n'$

$dup \bullet f = n \equiv f = n \bullet dup$

$\sum_i f = n_i \equiv \mathbf{true}$

Soundness: If $\vdash p \equiv q$, then $[\![p]\!] = [\![q]\!]$

Completeness: If $[\![p]\!] = [\![q]\!]$, then $\vdash p \equiv q$

Reduced NetKAT

Complete tests

α ::= **switch =** n • **port =** n

Complete assignments

β ::= **switch :=** n • **port :=** n

Reduced terms

```
p,q::= α          (* complete test *)
     | β          (* complete assignment *)
     | p + q      (* union *)
     | p • q      (* sequence *)
     | p*         (* Kleene star *)
     | dup        (* Duplication *)
```

Reduced NetKAT

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For simplicity, only consider two fields

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```

For simplicity, only consider two fields

Lemma: For every NetKAT term p, there is a reduced NetKAT term p' such that ⊢ p ≡ p'

Regular Interpretation

Can interpret reduced terms as regular languages over an "alphabet" of complete tests, complete assignments, and **dup**:

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Regular Interpretation: $R(p) \subseteq (A \cup B \cup \{\textbf{dup}\})^*$

$R(\alpha) = \{\alpha\}$

$R(\beta) = \{\beta\}$

$R(p + q) = R(p) \cup R(q)$

$R(p \bullet q) = R(p) \bullet R(q)$

$R(p^*) = R(p)^*$

$R(\textbf{dup}) = \{\textbf{dup}\}$

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Unfortunately $[\![p]\!] = [\![q]\!]$ does not imply $R(p) = R(q)$

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$R(\textbf{dup}) = \{\textbf{dup}\}$

Unfortunately $[\![p]\!] = [\![q]\!]$ does not imply $R(p) = R(q)$

Counterexample:

switch=1 · port=1 · switch=1 · port=2 and
switch=1 · port=1 · switch=2 · port=1

Language Model

Language Model

Language Interpretation: $G(p) \subseteq A \bullet (B \bullet \{\mathbf{dup}\})^* \bullet B$

$G(\alpha) = \{\alpha \bullet \pi_\alpha\}$

$G(\beta) = \{\alpha \bullet \beta \mid \alpha \in A\}$

$G(p + q) = G(p) \cup G(q)$

$G(p \bullet q) = G(p) \diamond G(q)$

$G(p^*) = G(p)^*$

$G(\mathbf{dup}) = \{\alpha \bullet \beta_\alpha \bullet \mathbf{dup} \bullet \beta_\alpha \mid \alpha \in A\}$

Language Model

Language Interpretation: $G(p) \subseteq A \bullet (B \bullet \{\mathbf{dup}\})^* \bullet B$

$G(\alpha) = \{\alpha \bullet \pi_\alpha\}$

$G(\beta) = \{\alpha \bullet \beta \mid \alpha \in A\}$

$G(p + q) = G(p) \cup G(q)$

$G(p \bullet q) = G(p) \diamond G(q)$

$G(p^*) = G(p)^*$

$G(\mathbf{dup}) = \{\alpha \bullet \beta_\alpha \bullet \mathbf{dup} \bullet \beta_\alpha \mid \alpha \in A\}$

Guarded strings

Language Model

Language Interpretation: $G(p) \subseteq A \bullet (B \bullet \{\mathbf{dup}\})^* \bullet B$

$G(a) = \{a \bullet \pi_a\}$

$G(\beta) = \{a \bullet \beta \mid a \in A\}$

$G(p + q) = G(p) \cup G(q)$

$G(p \bullet q) = G(p) \diamond G(q)$

$G(p^*) = G(p)^*$

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Guarded strings

Guarded concatenation

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Guarded strings

Guarded concatenation

Example: $\alpha_1 \bullet \beta_2 \bullet \mathbf{dup} \bullet \beta_3 \bullet \mathbf{dup} \bullet \ldots \bullet \mathbf{dup} \bullet \beta_n$

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Intuition: models trajectories through the network

Language Model

Language Interpretation: $G(p) \subseteq A \cdot (B \cdot \{\mathbf{dup}\})^* \cdot B$

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$G(p \cdot q) = G(p) \diamond G(q)$

$G(p^*) = G(p)^*$

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Guarded strings

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Example: $\alpha_1 \cdot \beta_2 \cdot \mathbf{dup} \cdot \beta_3 \cdot \mathbf{dup} \cdot \ldots \cdot \mathbf{dup} \cdot \beta_n$

Intuition: models trajectories through the network

Theorem: $[\![p]\!] = [\![q]\!]$ if and only if $G(p) = G(q)$

Language Model

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Intuition: models trajectories through the network

Theorem: $[\![p]\!] = [\![q]\!]$ if and only if $G(p) = G(q)$

Lemma: For all p, there exists a normal form $\hat{p}$ such that $\vdash p \equiv \hat{p}$ and $G(\hat{p})$ is regular

Completeness Proof

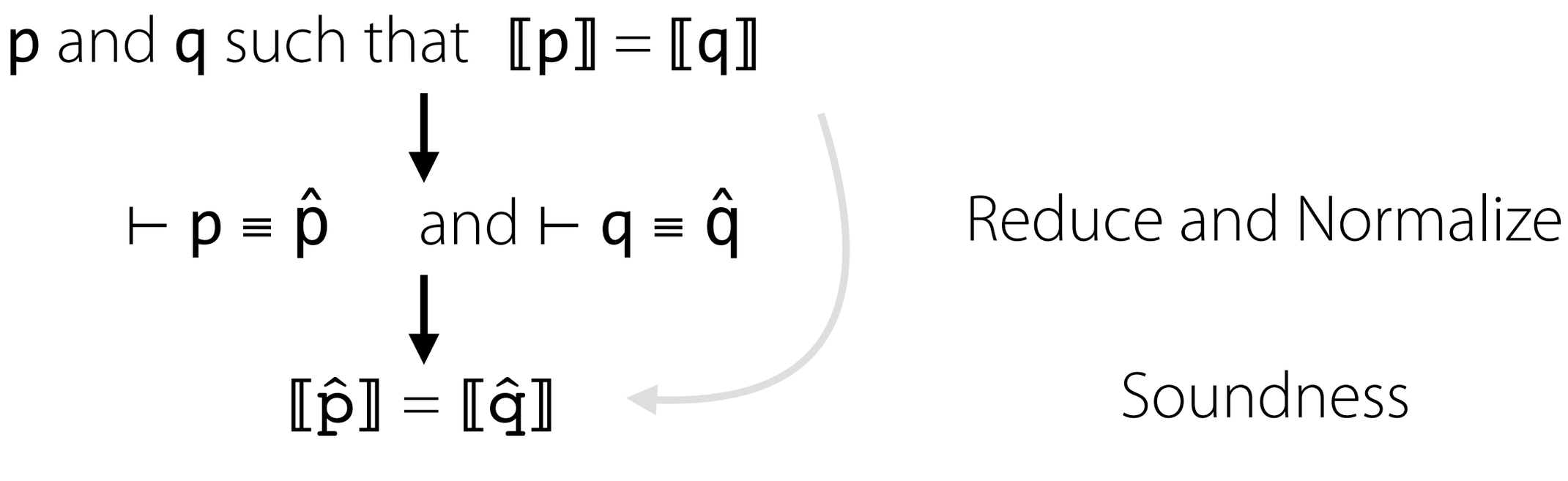
p and q such that $[\![\mathsf{p}]\!] = [\![\mathsf{q}]\!]$

Completeness Proof

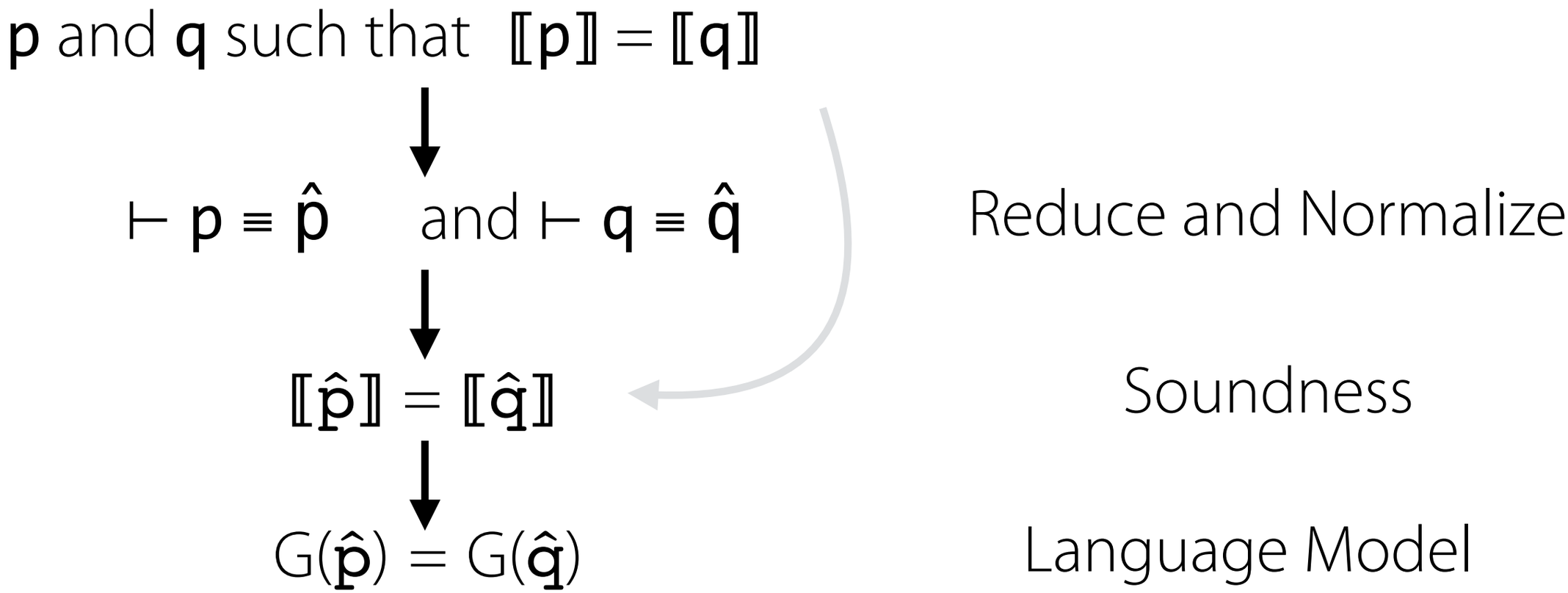
p and q such that $[\![\mathsf{p}]\!] = [\![\mathsf{q}]\!]$

$\vdash \mathsf{p} \equiv \hat{\mathsf{p}}$ and $\vdash \mathsf{q} \equiv \hat{\mathsf{q}}$ Reduce and Normalize

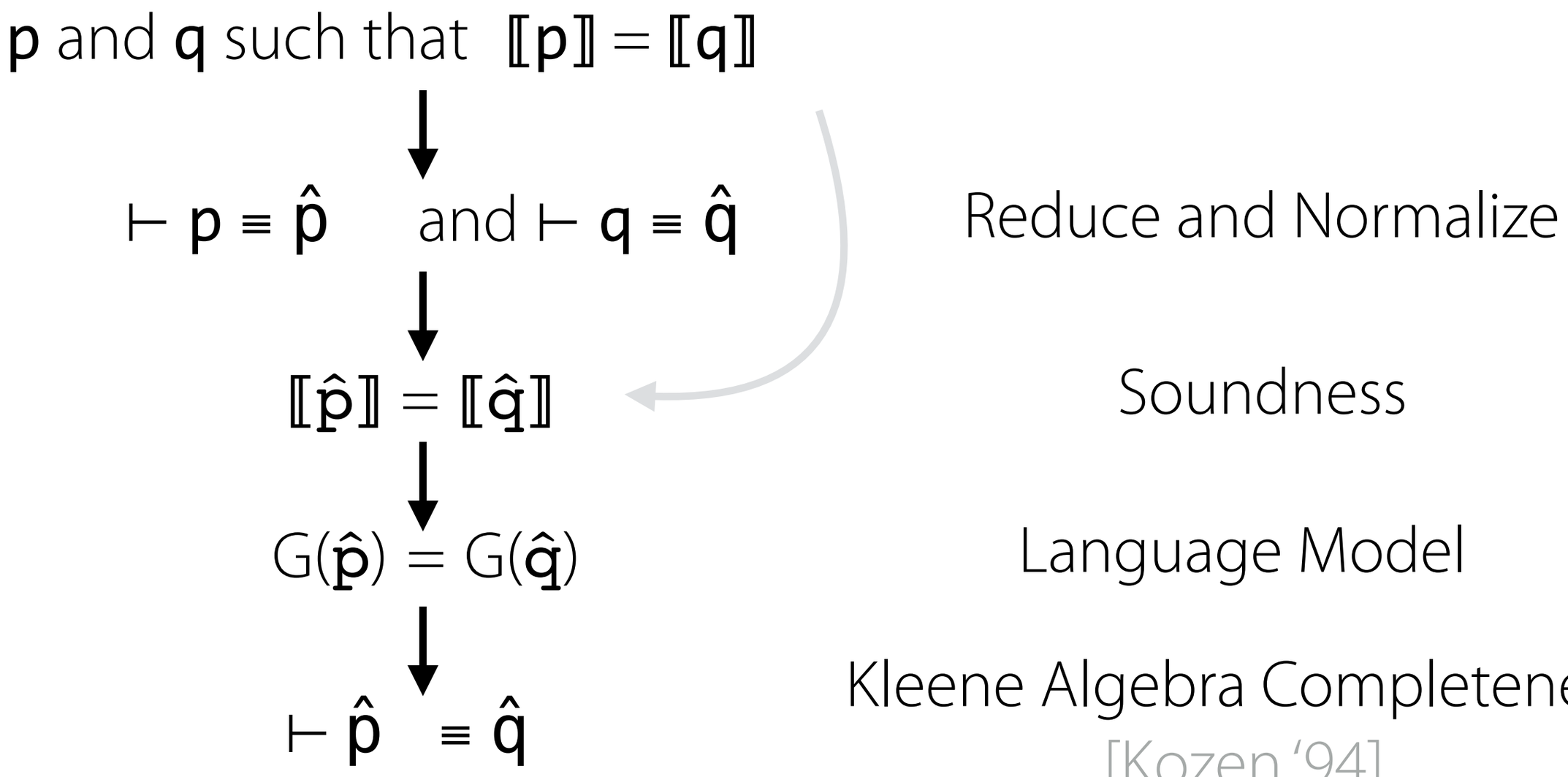
Completeness Proof



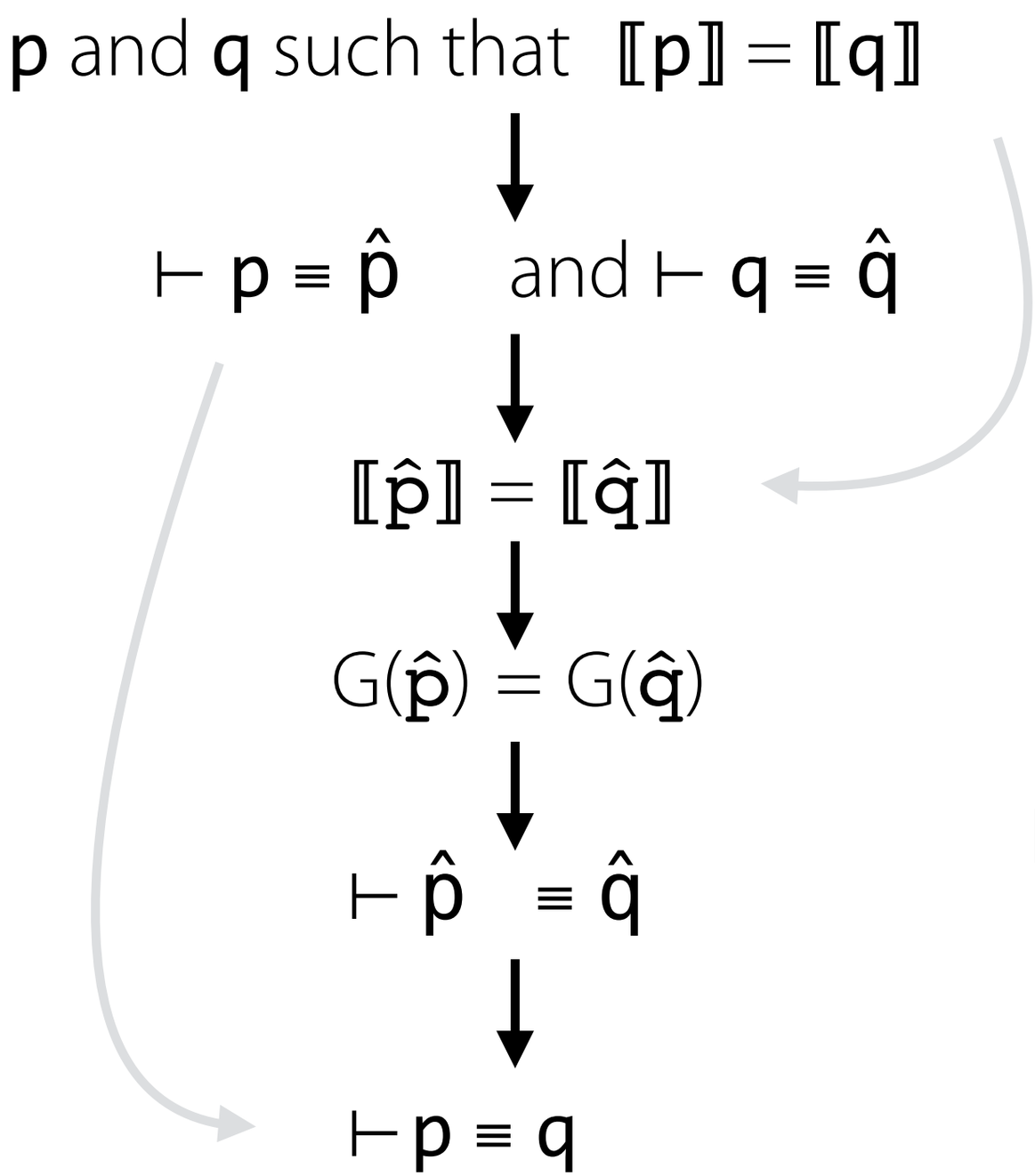
Completeness Proof



Completeness Proof



Completeness Proof



Reduce and Normalize

Soundness

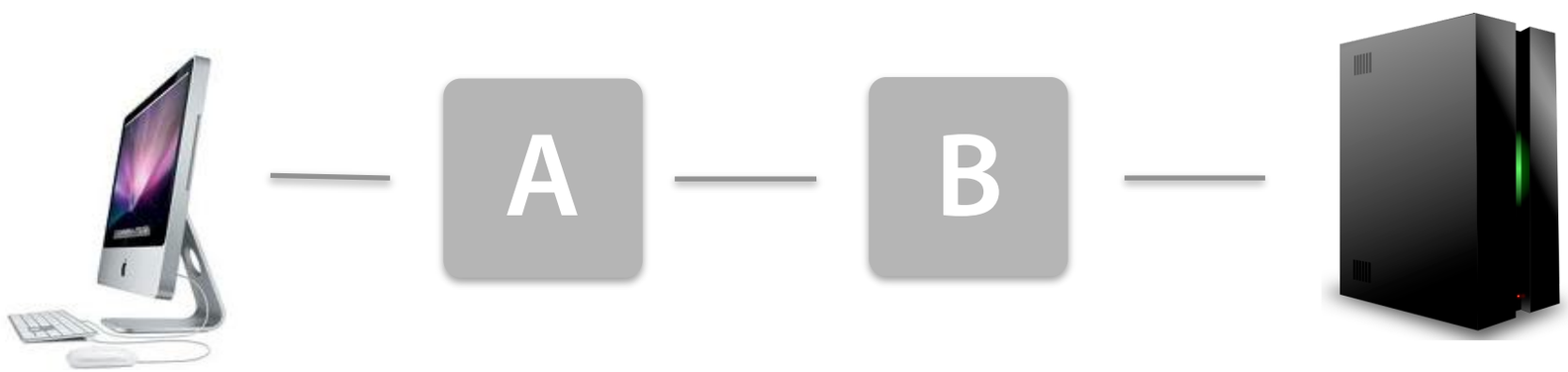
Language Model

Kleene Algebra Completeness
[Kozen '94]

Transitivity

Application: Optimization

Given a program and a topology:

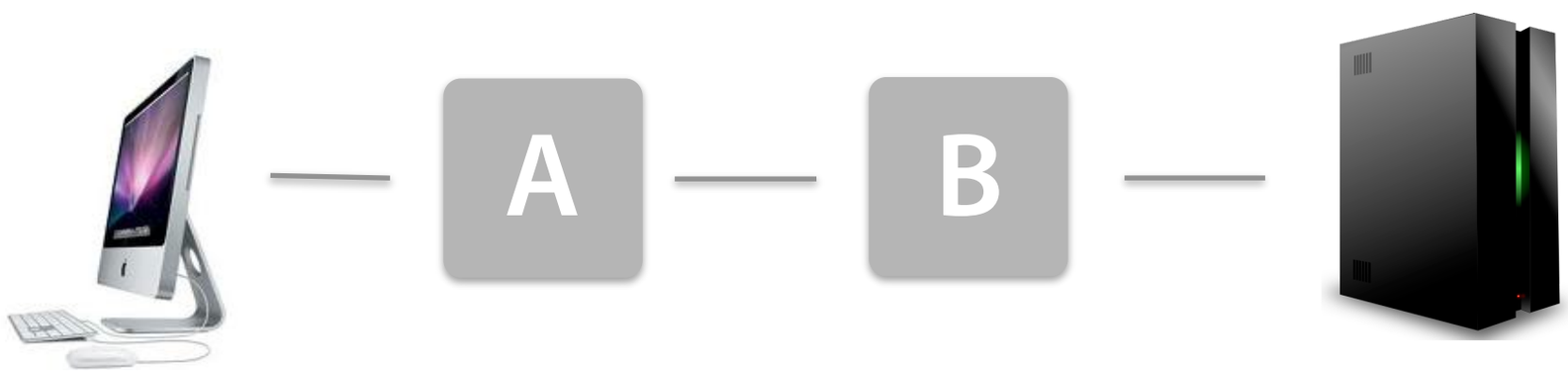


Want to be able to answer questions like:

> "Will my network behave the same if I put the firewall rules on A, or on switch B (or both)?"

Application: Optimization

Given a program and a topology:



Want to be able to answer questions like:

> "Will my network behave the same if I put the firewall rules on A, or on switch B (or both)?"

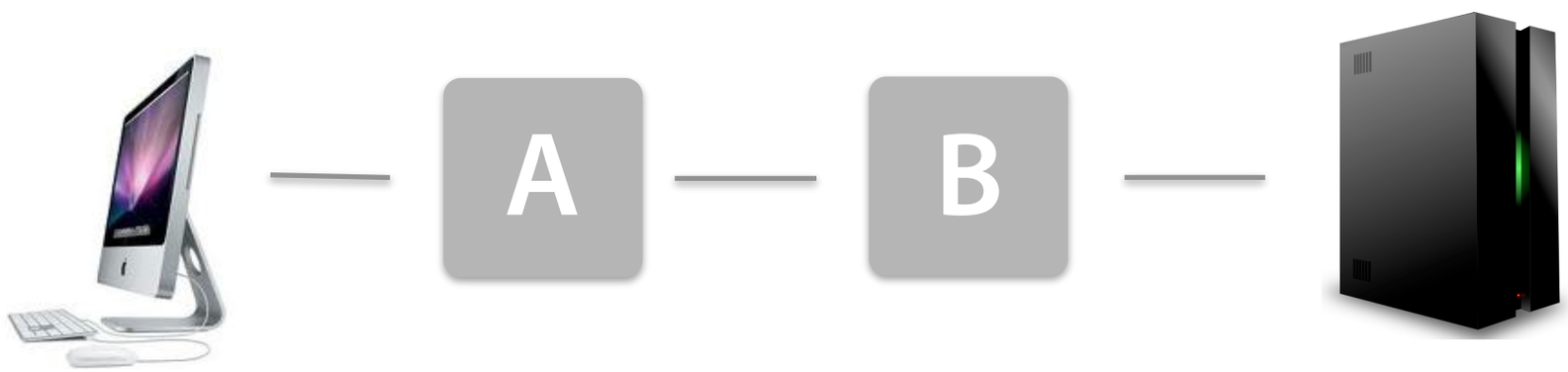
Formally, does the following equivalence hold?

$$(sw = A \bullet \underline{firewall} \bullet routing) + (sw = B \bullet routing)$$

$$\equiv$$

$$(sw = A \bullet routing) + (sw = B \bullet \underline{firewall} \bullet routing)$$

Code Motion Proof



$in \cdot \text{SSH} \cdot (p_A \cdot t)^* \cdot out$
$\equiv$ { KAT-Invariant, definition p_A }
$in \cdot \text{SSH} \cdot ((a_A \cdot \neg\text{SSH} \cdot p + a_B \cdot p) \cdot t \cdot \text{SSH})^* \cdot out$
$\equiv$ { KA-Seq-Dist-R }
$in \cdot \text{SSH} \cdot (a_A \cdot \neg\text{SSH} \cdot p \cdot t \cdot \text{SSH} + a_B \cdot p \cdot t \cdot \text{SSH})^* \cdot out$
$\equiv$ { KAT-Commute }
$in \cdot \text{SSH} \cdot (a_A \cdot \neg\text{SSH} \cdot \text{SSH} \cdot p \cdot t + a_B \cdot p \cdot t \cdot \text{SSH})^* \cdot out$
$\equiv$ { BA-Contra }
$in \cdot \text{SSH} \cdot (a_A \cdot 0 \cdot p \cdot t + a_B \cdot p \cdot t \cdot \text{SSH})^* \cdot out$
$\equiv$ { KA-Seq-Zero/Zero-Seq, KA-Plus-Comm, KA-Plus-Zero }
$in \cdot \text{SSH} \cdot (a_B \cdot p \cdot t \cdot \text{SSH})^* \cdot out$
$\equiv$ { KA-Unroll-L }
$in \cdot \text{SSH} \cdot (1 + (a_B \cdot p \cdot t \cdot \text{SSH}) \cdot (a_B \cdot p \cdot t \cdot \text{SSH})^*) \cdot out$
$\equiv$ { KA-Seq-Dist-L, KA-Seq-Dist-R, definition out }
$in \cdot \text{SSH} \cdot a_B \cdot a_2 +$
$in \cdot \text{SSH} \cdot a_B \cdot p \cdot t \cdot \text{SSH} \cdot (a_B \cdot p \cdot t \cdot \text{SSH})^* \cdot a_B \cdot a_2$
$\equiv$ { KAT-Commute }
$in \cdot a_B \cdot \text{SSH} \cdot a_2 +$
$in \cdot a_B \cdot \text{SSH} \cdot p \cdot t \cdot \text{SSH} \cdot (a_B \cdot p \cdot t \cdot \text{SSH})^* \cdot a_B \cdot a_2$
$\equiv$ { Lemma 1 }
$0 + 0$
$\equiv$ { KA-Plus-Idem }
0

$\equiv$ { KA-Plus-Idem }
$0 + 0$
$\equiv$ { Lemma 1, Lemma 2 }
$in \cdot a_B \cdot \text{SSH} \cdot a_2 +$
$in \cdot \text{SSH} \cdot (a_A \cdot p \cdot t \cdot \text{SSH})^* \cdot p \cdot \text{SSH} \cdot a_A \cdot t \cdot out$
$\equiv$ { KAT-Commute, definition out }
$in \cdot \text{SSH} \cdot out +$
$in \cdot \text{SSH} \cdot (a_A \cdot p \cdot t \cdot \text{SSH})^* \cdot a_A \cdot p \cdot t \cdot \text{SSH} \cdot out$
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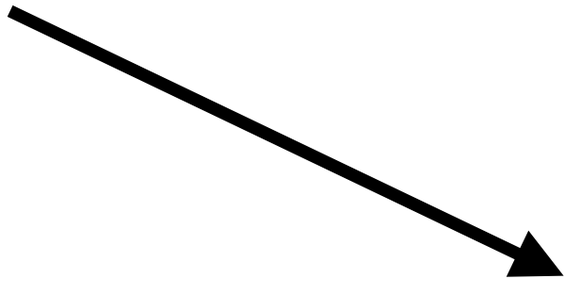
$\equiv$ { KAT-Invariant, definition p_B }
$in \cdot \text{SSH} \cdot (p_B \cdot t)^* \cdot out$

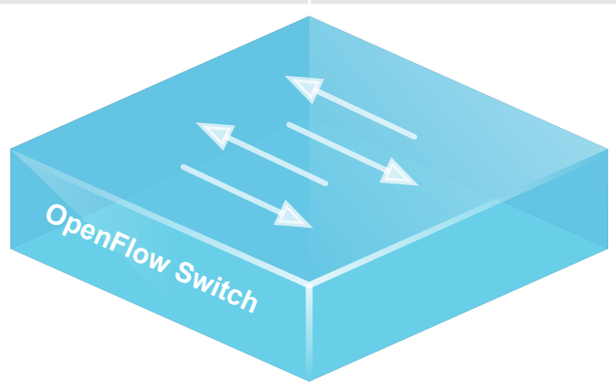
Application: Compiler Validation

```
!(type=SSH)·(pt=1·pt←2 + pt=2·pt←1)
```



OpenFlow rule table

Pattern	Action
type=SSH	Drop
port=1	Output 2
port=2	Output 1
*	Drop



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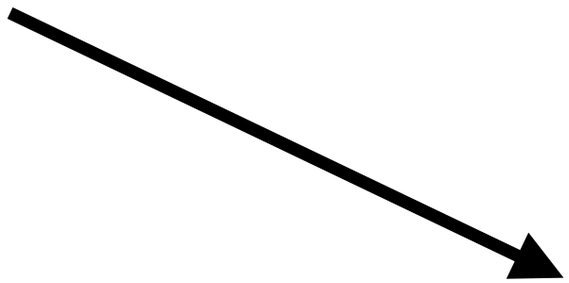
```
!(type=SSH)·(pt=1·pt←2 + pt=2·pt←1)
```



```
type=SSH·0 +
!type=SSH · pt=1·pt←2+
!type=SSH · !pt=1 · pt=2·pt←1+
!type=SSH · !pt=1 · !pt=2 · 0
```

OpenFlow rule table

Pattern	Action
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port=1	Output 2
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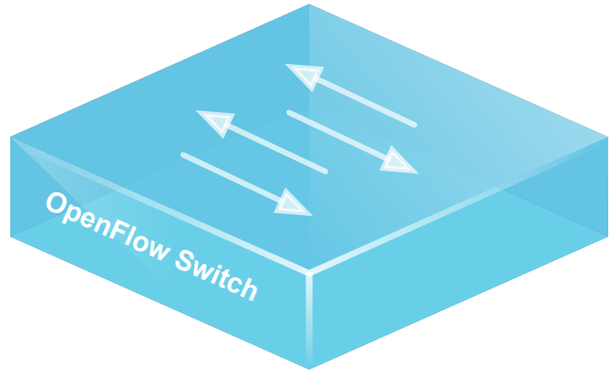
Theorem: p ≡ compile(p)



Operational Semantics

NetKAT Automata

A *NetKAT automaton* $M = (S, s_0, \varepsilon, \delta)$ is a tuple where:

- S is a finite set of states,
- $s_0 \in S$ is the start state,
- $\varepsilon \in S \to \text{Packet} \to \text{Packet Set}$ is the observation function
- $\delta \in S \to \text{Packet} \to (\text{State} * \text{Packet}) \text{ Set}$ is the continuation function

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Input "strings": $pkt_{in} \cdot pkt_1 \cdot \textbf{dup} \cdot \ldots \cdot \textbf{dup} \cdot pkt_n \; \textbf{dup} \cdot pkt_{out}$

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Input "strings": $\text{pkt}_{in} \bullet \text{pkt}_1 \bullet \textbf{dup} \bullet \ldots \bullet \textbf{dup} \bullet \text{pkt}_n \textbf{ dup} \bullet \text{pkt}_{out}$

A NetKAT automaton M *accepts* an input in state s if:

- accept s $(\text{pkt}_{in} \bullet \text{pkt}_{out}) \Leftrightarrow \text{pkt}_{out} \in \varepsilon\ s\ \text{pat}_{in}$
- accept s $(\text{pkt}_{in} \bullet \text{pkt}_1 \bullet \textbf{dup} \bullet w) \Leftrightarrow$

  $\exists\ s'.\ (\text{pkt}_1, s') \in \delta\ s\ \text{pkt}_{in}$ and accept s' $(\text{pkt}_1 \bullet w)$

NetKAT Derivatives

E(pol) ∈ Pol

D(pol) ∈ (Pol * L * Pol) Set

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Labels, one per occurrence of ***dup***

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E(pol) ∈ Pol

D(pol) ∈ (Pol * L * Pol) Set

Labels, one per occurrence of ***dup***

Observation map

E(**false**) = **false**

E(**true**) = **true**

E(f **=** n) = f **=** n

E(f **:=** n) = f **:=** n

E(**!**pol) = **!**pol

E(**dup**^{l}) = **false**

E(pol_1 **+** pol_2) = E(pol_1) **+** E(pol_2)

E(pol_1 • pol_2) = E(pol_1) • E(pol_2)

E(pol*) = E(pol)*

NetKAT Derivatives

E(pol) ∈ Pol

D(pol) ∈ (Pol * L * Pol) set

Continuation map

D(**false**) = {}

D(**true**) = {}

D(f**=**n) = {}

D(f**:=**n) = {}

D(**!**pol) = {}

D(**dup**^{l}) = { (**true**, l, **true**) }

D(pol_1 **+** pol_2) = D(pol_1) **+** D(pol_2)

D(pol_1 • pol_2) = D(pol_1) • pol_2 + E(pol_1) • D(pol_2)

D(pol*) = E(pol)* • D(pol) • pol*

NetKAT Derivatives

E(pol) ∈ Pol

D(pol) ∈ (Pol * L * Pol) set

Continuation map

D(**false**) = {}

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D(pol*) = E(pol)* • D(pol) • pol*

Theorem [POPL '15]: every NetKAT program has a finite number of derivatives (modulo ACI)

Program Automaton

Notation: Write k_l for the unique continuation of dup^l

The *program automaton* M for a program pol is a tuple where:

- S is the set of labels occurring in pol, plus the initial label 0
- $s_0 \triangleq 0$ is the start state,
- $\varepsilon\ l\ pkt = \{\ pkt' \mid \langle pkt' \rangle \in [\![E(k_l)]\!] \langle pkt \rangle\ \}$
- $\delta\ l\ pkt = \{\ (pkt', l') \mid (d, l', k) \in D(k_l) \text{ and } \langle pkt' \rangle \in [\![d]\!] \langle pkt \rangle\ \}$

Program Automaton

Lemma: For every label l occurring in a NetKAT program pol there exist unique programs d and k such that (d,l,k) ∈ D(pol).

Notation: Write k_l for the unique continuation of dup^l

The *program automaton* M for a program pol is a tuple where:

- S is the set of labels occurring in pol, plus the initial label 0
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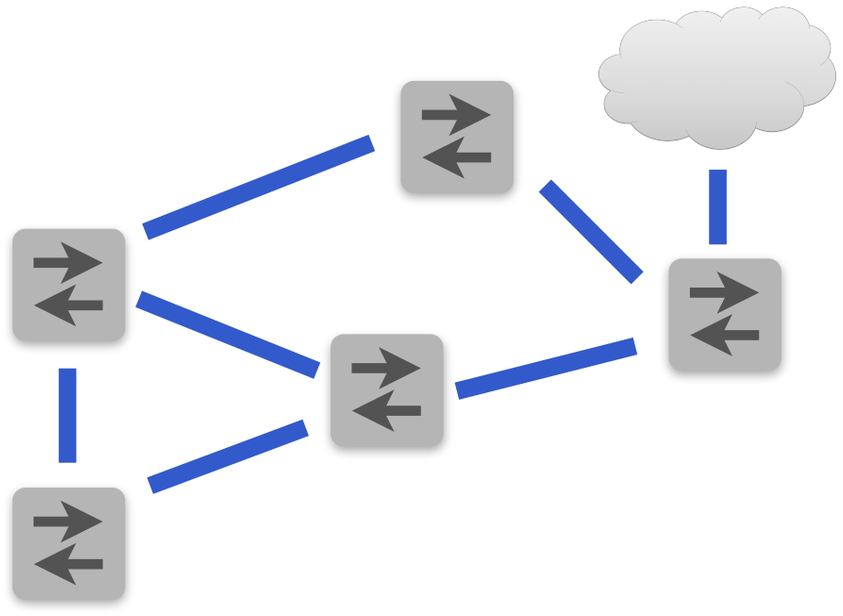
Application: Loop Freedom



Can also exploit automata representations to *efficiently* check whether a network is free of forwarding loops...

Intuition: $\forall \alpha.\ in \bullet (p \bullet t)^* \bullet \alpha \bullet (p \bullet t)^* \bullet \alpha \equiv$ **false**

- $\forall$ pkt, pkt′. pkt′ $\in [\![E(\Phi(in \bullet (p \bullet t)^*))]\!]$ pkt
- Check whether pkt′ $\in [\![E(\Phi(p \bullet t)^*))]\!]$ pkt′

Limits and Open Questions

Open Questions

The Horn theory of a KA/T is undecidable in general.
What theories are useful in practice?

How can we extend KA/T to model concurrency?

- Concurrent KA [CONCUR '09]
- KA with Observations [CONCUR '19]
- Concurrent KA with Observations [FoSSaCS '19]

Come to Tobias's POPL talk on Wednesday at 4:18pm!

Guarded Kleene Algebra with Tests

Reading

- Carolyn Jane Anderson, Nate Foster, Arjun Guha, Jean-Baptiste Jeannin, Dexter Kozen, Cole Schlesinger, and David Walker. **NetKAT: Semantic Foundations for Networks**. In *ACM SIGPLAN--SIGACT Symposium on Principles of Programming Languages (POPL),* January 2014.
- Steffen Smolka, Spiridon Eliopoulos, Nate Foster, and Arjun Guha. **A Fast Compiler for NetKAT.** *In ACM SIGPLAN International Conference on Functional Programming (ICFP)*, September 2015.
- Nate Foster, Dexter Kozen, Matthew Milano, Alexandra Silva, and Laure Thompson. **A Coalgebraic Decision Procedure for NetKAT.** In ACM SIGPLAN--SIGACT Symposium on Principles of Programming Languages (POPL), Mumbai, India, January 2015.