On Moessner's theorem

Dexter Kozen[1] Alexandra Silva[2,3]

[1]Cornell University, USA
[2]Radboud Universiteit Nijmegen, The Netherlands
[3]HasLab, Universidade do Minho, Portugal

QAIS workshop, October 2011

Moessner's construction (n=4)

1	2	3	~~4~~	5	6	7	~~8~~	9	10	11	~~12~~	13	14	15	~~16~~	17	18	19	~~20~~	21	22	23	~~24~~
1	3	~~6~~		11	17	~~24~~		33	43	~~54~~		67	81	~~96~~		113	131	~~150~~		171	193	~~216~~	
1	~~4~~			15	~~32~~			65	~~108~~			175	~~256~~			369	~~500~~			671	~~864~~		
1				16				81				256				625				1296			
1^4				2^4				3^4				4^4				5^4				6^4			

Moessner's Conjecture/Theorem

Works for all $n \in \mathbb{N}$

Moessner's construction (n=4)

1	2	3	~~4~~	5	6	7	~~8~~	9	10	11	~~12~~	13	14	15	~~16~~	17	18	19	~~20~~	21	22	23	~~24~~
1	3	~~6~~		11	17	~~24~~		33	43	~~54~~		67	81	~~96~~		113	131	~~150~~		171	193	~~216~~	
1	~~4~~			15	~~32~~			65	~~108~~			175	~~256~~			369	~~500~~			671	~~864~~		
1				16				81				256				625				1296			

1^4 2^4 3^4 4^4 5^4 6^4

Moessner's Conjecture/Theorem

Works for all $n \in \mathbb{N}$

Moessner's construction (n=4)

1	2	3	~~4~~	5	6	7	~~8~~	9	10	11	~~12~~	13	14	15	~~16~~	17	18	19	~~20~~	21	22	23	~~24~~
1	3	~~6~~		11	17	~~24~~		33	43	~~54~~		67	81	~~96~~		113	131	~~150~~		171	193	~~216~~	
1	~~4~~			15	~~32~~			65	~~108~~			175	~~256~~			369	~~500~~			671	~~864~~		
1				16				81				256				625				1296			

1^4 2^4 3^4 4^4 5^4 6^4

Moessner's Conjecture/Theorem

Works for all $n \in \mathbb{N}$

Moessner's construction (n=4)

1	2	3	~~4~~	5	6	7	~~8~~	9	10	11	~~12~~	13	14	15	~~16~~	17	18	19	~~20~~	21	22	23	~~24~~
1	3	~~6~~		11	17	~~24~~		33	43	~~54~~		67	81	~~96~~		113	131	~~150~~		171	193	~~216~~	
1	~~4~~			15	~~32~~			65	~~108~~			175	~~256~~			369	~~500~~			671	~~864~~		
1				16				81				256				625				1296			
1^4				2^4				3^4				4^4				5^4				6^4			

Moessner's Conjecture/Theorem

Works for all $n \in \mathbb{N}$

Moessner's construction (n=4)

1	2	3	~~4~~	5	6	7	~~8~~	9	10	11	~~12~~	13	14	15	~~16~~	17	18	19	~~20~~	21	22	23	~~24~~
1	3	~~6~~		11	17	~~24~~		33	43	~~54~~		67	81	~~96~~		113	131	~~150~~		171	193	~~216~~	
1	~~4~~			15	~~32~~			65	~~108~~			175	~~256~~			369	~~500~~			671	~~864~~		
1				16				81				256				625				1296			
1^4				2^4				3^4				4^4				5^4				6^4			

Moessner's Conjecture/Theorem

Works for all $n \in \mathbb{N}$

Moessner's construction (n=4)

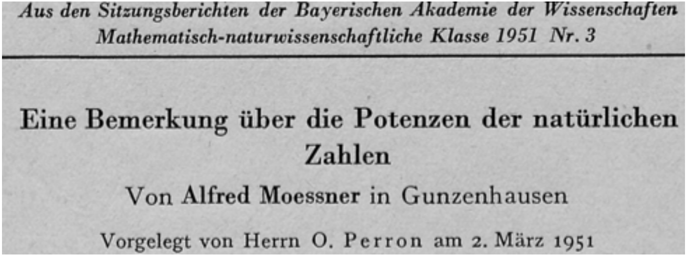
1	2	3	~~4~~	5	6	7	~~8~~	9	10	11	~~12~~	13	14	15	~~16~~	17	18	19	~~20~~	21	22	23	~~24~~
1	3	~~6~~		11	17	~~24~~		33	43	~~54~~		67	81	~~96~~		113	131	~~150~~		171	193	~~216~~	
1	~~4~~			15	~~32~~			65	~~108~~			175	~~256~~			369	~~500~~			671	~~864~~		
1				16				81				256				625				1296			
1^4				2^4				3^4				4^4				5^4				6^4			

Moessner's Conjecture/Theorem

Works for all $n \in \mathbb{N}$

History

1951 Moessner conjectures it


Aus den Sitzungsberichten der Bayerischen Akademie der Wissenschaften
Mathematisch-naturwissenschaftliche Klasse 1951 Nr. 3

Eine Bemerkung über die Potenzen der natürlichen Zahlen

Von Alfred Moessner in Gunzenhausen

Vorgelegt von Herrn O. Perron am 2. März 1951

1952 Perron proves it

1952 Paasche and Salie generalize it

1966 Long presents and alternative proof (and generalizes it)

2010 Hinze, Rutten&Niqui present new proofs of Moessner's theorem

2011 This talk: an uniform proof of all the theorems

History

1951 Moessner conjectures it

Aus den Sitzungsberichten der Bayerischen Akademie der Wissenschaften
Mathematisch-naturwissenschaftliche Klasse 1951 Nr. 3

Eine Bemerkung über die Potenzen der natürlichen Zahlen

Von Alfred Moessner in Gunzenhausen

Vorgelegt von Herrn O. Perron am 2. März 1951

1952 Perron proves it

1952 Paasche and Salie generalize it

1966 Long presents and alternative proof (and generalizes it)

2010 Hinze, Rutten&Niqui present new proofs of Moessner's theorem

2011 This talk: an uniform proof of all the theorems

History

1951 Moessner conjectures it

Aus den Sitzungsberichten der Bayerischen Akademie der Wissenschaften
Mathematisch-naturwissenschaftliche Klasse 1951 Nr. 3

Eine Bemerkung über die Potenzen der natürlichen Zahlen

Von Alfred Moessner in Gunzenhausen

Vorgelegt von Herrn O. Perron am 2. März 1951

1952 Perron proves it

1952 Paasche and Salie generalize it

1966 Long presents and alternative proof (and generalizes it)

2010 Hinze, Rutten&Niqui present new proofs of Moessner's theorem

2011 This talk: an uniform proof of all the theorems

But first…

Paasche asked: what if we cross out 1, 3, 6, 10, …?

~~1~~	2	~~3~~	4	5	~~6~~	7	8	9	~~10~~	11	12	13	14	~~15~~	16	17	18	19	20	~~21~~
	2		6	~~11~~		18	26	~~35~~		46	58	71	~~85~~		101	118	136	155	~~175~~	
			6			24	~~50~~			96	154	~~225~~			326	444	580	~~735~~		
						24				120	~~274~~				600	1044	~~1624~~			
										120					720	~~1764~~				
															720					

We get the factorials: 1, 2, 6, 24, 120, … = 1!, 2!, 3!, 4!, 5!, … .

But first. . .

Paasche asked: what if we cross out 1, 3, 6, 10, . . . ?

~~1~~	2	~~3~~	4	5	~~6~~	7	8	9	~~10~~	11	12	13	14	~~15~~	16	17	18	19	20	~~21~~
	2		6	~~11~~		18	26	~~35~~		46	58	71	~~85~~		101	118	136	155	~~175~~	
			6			24	~~50~~			96	154	~~225~~			326	444	580	~~735~~		
						24				120	~~274~~				600	1044	~~1624~~			
										120					720	~~1764~~				
															720					

We get the factorials: 1, 2, 6, 24, 120, . . . = 1!, 2!, 3!, 4!, 5!, ...

But first...

Paasche asked: what if we cross out 1, 3, 6, 10, ...?

~~1~~	2	~~3~~	4	5	~~6~~	7	8	9	~~10~~	11	12	13	14	~~15~~	16	17	18	19	20	~~21~~
	2		6	~~11~~		18	26	~~35~~		46	58	71	~~85~~		101	118	136	155	~~175~~	
			6			24	~~50~~			96	154	~~225~~			326	444	580	~~735~~		
						24				120	~~274~~				600	1044	~~1624~~			
										120					720	~~1764~~				
															720					

We get the factorials: 1, 2, 6, 24, 120, ... = 1!, 2!, 3!, 4!, 5!, ...

And the magic continues...

What if we increment the increment by one in each step?

~~1~~	2	3	~~4~~	5	6	7	8	9	~~10~~	11	12	13	14	15	16	17	18	19	~~20~~	...
	2	~~5~~		10	16	23	31	~~40~~		51	63	76	80	95 ...						
	2			12	28	51	~~82~~			133	196	272	352 ...							
				12	40	~~91~~				224	420	692 ...								
				12	~~52~~					276	696 ...									
				12						288	984 ...									
										288	1272 ...									
										288 ...										
										288 ...										
										288 ...										

What is the sequence

$$1, 2, 12, 288, \ldots ?$$

It's the superfactorials!

$$1, 2, 12, 288, \ldots = 1!, 2!1!, 3!2!1!, 4!3!2!1!, \ldots = 1!!, 2!!, 3!!, 4!!, \ldots .$$

And the magic continues...

What if we increment the increment by one in each step?

~~1~~	2	3	~~4~~	5	6	7	8	9	~~10~~	11	12	13	14	15	16	17	18	19	~~20~~	...
	2	~~5~~		10	16	23	31	~~40~~		51	63	76	80	95	...					
	2			12	28	51	~~82~~			133	196	272	352	...						
				12	40	~~91~~				224	420	692	...							
				12	~~52~~					276	696	...								
				12						288	984	...								
										288	1272	...								
										288	...									
										288	...									
										288	...									

What is the sequence

$$1, 2, 12, 288, \ldots ?$$

It's the superfactorials!

$$1, 2, 12, 288, \ldots = 1!, 2!1!, 3!2!1!, 4!3!2!1!, \ldots = 1!!, 2!!, 3!!, 4!!, \ldots .$$

And the magic continues. . .

What if we increment the increment by one in each step?

```
1̶   2   3   4̶   5   6   7   8   9  1̶0̶  11  12  13  14  15  16  17  18  19  2̶0̶ ...
    2   5̶       10  16  23  31  4̶0̶      51  63  76  80  95 ...
    2           12  28  51  8̶2̶         133 196 272 352 ...
                12  40  9̶1̶             224 420 692 ...
                12  5̶2̶                 276 696 ...
                12                     288 984 ...
                                       288 1272 ...
                                       288 ...
                                       288 ...
                                       288 ...
```

What is the sequence

$$1, 2, 12, 288, \ldots ?$$

It's the superfactorials!

$$1, 2, 12, 288, \ldots = 1!, 2!1!, 3!2!1!, 4!3!2!1!, \ldots = 1!!, 2!!, 3!!, 4!!, \ldots .$$

And the magic continues. . .

What if we increment the increment by one in each step?

~~1~~	2	3	~~4~~	5	6	7	8	9	~~10~~	11	12	13	14	15	16	17	18	19	~~20~~	...
	2	~~5~~		10	16	23	31	~~40~~		51	63	76	80	95	...					
	2			12	28	51	~~82~~			133	196	272	352	...						
				12	40	~~91~~				224	420	692	...							
				12	~~52~~					276	696	...								
				12						288	984	...								
										288	1272	...								
										288	...									
										288	...									
										288	...									

What is the sequence

$$1, 2, 12, 288, \ldots ?$$

It's the superfactorials!

$$1, 2, 12, 288, \ldots = 1!, 2!1!, 3!2!1!, 4!3!2!1!, \ldots = 1!!, 2!!, 3!!, 4!!, \ldots .$$

An alternative procedure

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	3	6		11	17	24		33	43	54		67	81	96	
1	4			15	32			65	108			175	256		
1				16				81				256			

Long's observation

First triangle: Pascal triangle; all have Pascal property

Long's procedure

starting point: Pascal triangle
next step: consider the *n*th northeast-to-southwest row. Take prefix sums and make that the first column, and let the first row be a sequence of 1's. Complete the triangle using the Pascal property.

An alternative procedure

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	2	3	4		5	6	7	8		9	10	11	12		13	14	15	16	
1	3	6			11	17	24			33	43	54			67	81	96		
1	4				15	32				65	108				175	256			
1					16					81					256				

Long's observation

First triangle: Pascal triangle; all have Pascal property

Long's procedure

starting point: Pascal triangle
next step: consider the *n*th northeast-to-southwest row. Take prefix sums and make that the first column, and let the first row be a sequence of 1's. Complete the triangle using the Pascal property.

An alternative procedure

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	2	3	4		5	6	7	8		9	10	11	12		13	14	15	16	
1	3	6			11	17	24			33	43	54			67	81	96		
1	4				15	32				65	108				175	256			
1					16					81					256				

Long's observation

First triangle: Pascal triangle; all have Pascal property

Long's procedure

starting point: Pascal triangle
next step: consider the *n*th northeast-to-southwest row. Take prefix sums and make that the first column, and let the first row be a sequence of 1's. Complete the triangle using the Pascal property.

An alternative procedure

1	1	1	1	1		1	1	1	1	1		1	1	1	1	1		1	1	1	1	1
1	2	3	4			5	6	7	8			9	10	11	12			13	14	15	16	
1	3	6				11	17	24				33	43	54				67	81	96		
1	4					15	32					65	108					175	256			
1						16						81						256				

Long's observation

First triangle: Pascal triangle; all have Pascal property

Long's procedure

starting point: Pascal triangle
next step: consider the *n*th northeast-to-southwest row. Take prefix sums and make that the first column, and let the first row be a sequence of 1's.
Complete the triangle using the Pascal property.

Yet another generalization (Long & Salie)

What if instead of the natural numbers we start with

a	a+d	a+2d	a+3d	a+4d	a+5d	a+6d	~~a+7d~~	a+8d	a+9d	a+10d	~~a+11d~~

a	a+d	a+2d	~~a+3d~~	a+4d	a+5d	a+6d	~~a+7d~~	a+8d	a+9d	a+10d	~~a+11d~~
a	2a+d	~~3a+3d~~		4a+7d	5a+12d	~~6a+18d~~		7a+26d	8a+35d	9a+45d	
a	~~3a+d~~			7a+8d	~~12a+20d~~			19a+46d	~~27a+81d~~		
a				8a+8d				27a+54d			
a			(a+d).8				(a+2d).27				

Given $n \in \mathbb{N}$, the Moessner construction yields

$$a \cdot 1^{n-1}, (a+d) \cdot 2^{n-1}, (a+2d) \cdot 3^{n-1}, \ldots$$

when starting from the sequence $a, a+d, a+2d, \ldots$

Yet another generalization (Long & Salie)

What if instead of the natural numbers we start with

a	a+d	a+2d	a+3d	a+4d	a+5d	a+6d	~~a+7d~~	a+8d	a+9d	a+10d	~~a+11d~~

a	a+d	a+2d	~~a+3d~~	a+4d	a+5d	a+6d	~~a+7d~~	a+8d	a+9d	a+10d	~~a+11d~~
a	2a+d	~~3a+3d~~		4a+7d	5a+12d	~~6a+18d~~		7a+26d	8a+35d	9a+45d	
a	~~3a+d~~			7a+8d	~~12a+20d~~			19a+46d	~~27a+81d~~		
a				8a+8d				27a+54d			
a				(a+d).8				(a+2d).27			

Given $n \in \mathbb{N}$, the Moessner construction yields

$$a \cdot 1^{n-1}, (a+d) \cdot 2^{n-1}, (a+2d) \cdot 3^{n-1}, \ldots$$

when starting from the sequence $a, a+d, a+2d, \ldots$

Yet another generalization (Long & Salie)

What if instead of the natural numbers we start with

a	a+d	a+2d	a+3d	a+4d	a+5d	a+6d	~~a+7d~~	a+8d	a+9d	a+10d	~~a+11d~~

a	a+d	a+2d	~~a+3d~~	a+4d	a+5d	a+6d	~~a+7d~~	a+8d	a+9d	a+10d	~~a+11d~~
a	2a+d	~~3a+3d~~		4a+7d	5a+12d	~~6a+18d~~		7a+26d	8a+35d	9a+45d	
a	~~3a+d~~			7a+8d	~~12a+20d~~			19a+46d	~~27a+81d~~		
a				8a+8d				27a+54d			
a				(a+d).8				(a+2d).27			

Given $n \in \mathbb{N}$, the Moessner construction yields

$$a \cdot 1^{n-1}, (a+d) \cdot 2^{n-1}, (a+2d) \cdot 3^{n-1}, \ldots$$

when starting from the sequence $a, a+d, a+2d, \ldots$

Proofs

- The proofs of Perron, Paasche, Long and Salie all have in common the manipulation of binomial coefficients
- Hinze's proof (2010) involves calculations scans (FP)
- Rutten's proof is coinductive
- Not obvious if the last two can be generalized

Our view on Moessner's theorem

- we take Long's triangle view
- we describe the process as operations on formal power series on **two** variables
- this yields a proof of Moesser's theorem and its generalizations all at once!

Our view on Moessner's theorem

- we take Long's triangle view
- we describe the process as operations on formal power series on **two** variables
- this yields a proof of Moesser's theorem and its generalizations all at once!

Our view on Moessner's theorem

- we take Long's triangle view
- we describe the process as operations on formal power series on **two** variables
- this yields a proof of Moesser's theorem and its generalizations all at once!

Pascal triangle

The Pascal triangle $\Delta = \Delta(x, y)$ is

$$\Delta(x, y) = \frac{1}{1 - (x + y)} = \sum_{m=0}^{\infty} (x + y)^m = \sum_{i,j} \binom{i + j}{i} x^i y^j. \qquad (1)$$

Moessner's construction, algebraically

The "*n*th northeast-to-southwest row" of $p \in \mathbb{Z}(x, y)$ is the homogeneous component of degree n, denoted $[p]_n$.

The operation of "taking prefix sums" is multiplying by $\frac{1}{1-x} = \sum_{i=0}^{\infty} x^i$.

Sequences of triangles

Each successive level-n Moessner triangle is obtained from the previous by taking the homogeneous component of degree n, evaluating at $y = 1$, and multiplying by Δ.

We define inductively

$$h_0(x,y) = 1 \qquad\qquad h_{k+1}(x,y) = [h_k(x,1)\cdot\Delta(x,y)]_n\,,$$

then the kth level-n Moessner triangle is $h_k(x,1)\cdot\Delta$ and the final sequence in the Moessner construction is the lead coefficient of $h_k(x,1)$ for $k = 1,2,3,\ldots$.

Sequence of triangles generalized

Instead of $h_0(x,y) = 1$, we can take $h_0 \in \mathbb{Z}[x,y]$ arbitrary (Salie's generalization).

For Paasche's contruction we need to take homogeneous components not of a fixed n but of an arbitrary increasing sequence.

Let $d(0), d(1), d(2), \ldots$ of nonnegative integers and $n(k) = \sum_{i=0}^{k} d(i)$. The $n(k)$'s are the positions one should delete.

For Moessner

$$\begin{array}{lllll} d(0) & d(1) & d(2) & d(3) & \cdots \\ n & 0 & 0 & 0 & \cdots \end{array}$$

$$\begin{array}{lllll} n & n & n & n & \cdots \\ n(0) & n(1) & n(2) & n(3) & \cdots \end{array}$$

Sequence of triangles generalized

Instead of $h_0(x, y) = 1$, we can take $h_0 \in \mathbb{Z}[x, y]$ arbitrary (Salie's generalization).

For Paasche's contruction we need to take homogeneous components not of a fixed n but of an arbitrary increasing sequence.

Let $d(0), d(1), d(2), \ldots$ of nonnegative integers and $n(k) = \sum_{i=0}^{k} d(i)$. The $n(k)$'s are the positions one should delete.

For Moessner

$$\begin{array}{lllll} d(0) & d(1) & d(2) & d(3) & \cdots \\ n & 0 & 0 & 0 & \cdots \end{array}$$

$$\begin{array}{lllll} n & n & n & n & \cdots \\ n(0) & n(1) & n(2) & n(3) & \cdots \end{array}$$

Sequence of triangles generalized

Instead of $h_0(x, y) = 1$, we can take $h_0 \in \mathbb{Z}[\,x, y\,]$ arbitrary (Salie's generalization).

For Paasche's contruction we need to take homogeneous components not of a fixed n but of an arbitrary increasing sequence.

Let $d(0), d(1), d(2), \ldots$ of nonnegative integers and $n(k) = \sum_{i=0}^{k} d(i)$. The $n(k)$'s are the positions one should delete.

For Moessner

$$\begin{array}{lllll} d(0) & d(1) & d(2) & d(3) & \cdots \\ n & 0 & 0 & 0 & \cdots \\ & & & & \\ n & n & n & n & \cdots \\ n(0) & n(1) & n(2) & n(3) & \cdots \end{array}$$

Sequence of triangles generalized

Instead of $h_0(x,y) = 1$, we can take $h_0 \in \mathbb{Z}[x,y]$ arbitrary (Salie's generalization).

For Paasche's contruction we need to take homogeneous components not of a fixed n but of an arbitrary increasing sequence.

Let $d(0), d(1), d(2), \ldots$ of nonnegative integers and $n(k) = \sum_{i=0}^{k} d(i)$. The $n(k)$'s are the positions one should delete.

For Moessner

$$\begin{array}{lllll} d(0) & d(1) & d(2) & d(3) & \cdots \\ n & 0 & 0 & 0 & \cdots \end{array}$$

$$\begin{array}{lllll} n & n & n & n & \cdots \\ n(0) & n(1) & n(2) & n(3) & \cdots \end{array}$$

Sequence of triangles generalized II

Define inductively

$$h_{k+1}(x,y) = [h_k(x,1) \cdot \Delta(x,y)]_{n(k+1)} \,. \tag{2}$$

The Moessner construction is the special case $h_0 = 1$, $d(0) = n$, and $d(i) = 0$ for $i \geq 1$.

Main theorem

Theorem

Let h_k be the sequence defined by (2). For all $k \geq 0$,

$$h_k(x, y) = \prod_{i=0}^{k-1} ((k-i)x + y)^{d(i)} \cdot h_0(x, kx + y).$$

Corollaries

Paasche's, Long's, and Moessner's theorems are now immediate consequences of Theorem 1.

Corollary (Moessner's Theorem)

If $h_0 = 1$, $d(0) = n$, and $d(k) = 0$ for $k \geq 1$, then the lead coefficient of $h_k(x, 1)$ is k^n for all $k \geq 1$.

Corollaries

Paasche's, Long's, and Moessner's theorems are now immediate consequences of Theorem 1.

Corollary (Long's Theorem)

If $h_0 = (a-d)x + dy$, $d(0) = n-1$, and $d(k) = 0$ for $k \geq 1$, then the lead coefficient of $h_k(x,1)$ is $(a + (k-1)d)k^{n-1}$ for all $k \geq 1$.

Paasche's, Long's, and Moessner's theorems are now immediate consequences of Theorem 1.

Corollary (Paasche's Theorem)

For $h_0 = 1$ and any sequence d, the lead coefficient of $h_k(x, 1)$ is

$$\prod_{i=0}^{k-1} (k-i)^{d(i)} \tag{3}$$

for all $k \geq 0$. In particular, the sequences $d = 1, 1, 1, \ldots$ and $d = 1, 2, 3, \ldots$ yield the factorials and superfactorials, respectively.

Conclusions

- First proof that covers all the generalizations
- Proofs have a striking simplicity (no binomial coefficent manipulations!)
- Opens the door to new Moessner-like theorems (multidimensional generalization).

Conclusions

- First proof that covers all the generalizations
- Proofs have a striking simplicity (no binomial coefficent manipulations!)
- Opens the door to new Moessner-like theorems (multidimensional generalization).

Conclusions

- First proof that covers all the generalizations
- Proofs have a striking simplicity (no binomial coefficent manipulations!)
- Opens the door to new Moessner-like theorems (multidimensional generalization).

Thank you for your attention!