Sound and Complete axiomatization of trace semantics for probabilistic systems

Alexandra Silva[1] Ana Sokolova[2]

[1]Centrum Wiskunde & Informatica (currently on leave at Cornell Univ.)
[2]Computer Sciences Department, University of Salzburg

June 2011

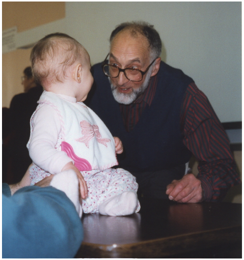
Motivation

$P ::= \mathbf{0} \mid a.P \mid P + P \mid \mu x.P^g$

Kleene-like theorem: behaviours of LTS are characterized by P's and vice-versa

Axiomatization:
$P + Q \equiv Q + P; P + \mathbf{0} \equiv P; \mu x.P \equiv P[\mu x.P/x]; \ldots$

Soundness and Completeness:
$P \equiv Q \iff P \sim Q$



Robin Milner

Robin Milner: A Complete Inference System for a Class of Regular Behaviours. J. Comput. Syst. Sci. 28(3): 439-466 (1984)

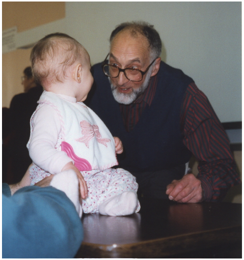
Motivation

$P ::= \mathbf{0} \mid a.P \mid P + P \mid \mu x.P^g$

Kleene-like theorem: behaviours of LTS are characterized by P's and vice-versa

Axiomatization:
$P + Q \equiv Q + P; P + \mathbf{0} \equiv P; \mu x.P \equiv P[\mu x.P/x]; \ldots$

Soundness and Completeness:
$P \equiv Q \iff P \sim Q$



Robin Milner

Robin Milner: A Complete Inference System for a Class of Regular Behaviours. J. Comput. Syst. Sci. 28(3): 439-466 (1984)

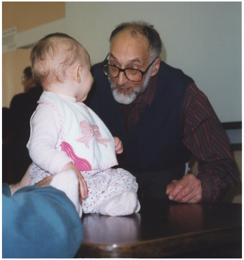
Motivation

$P ::= \mathbf{0} \mid a.P \mid P + P \mid \mu x.P^g$

Kleene-like theorem: behaviours of LTS are characterized by P's and vice-versa

Axiomatization:
$P + Q \equiv Q + P; P + \mathbf{0} \equiv P; \mu x.P \equiv P[\mu x.P/x]; \ldots$

Soundness and Completeness:
$P \equiv Q \iff P \sim Q$



Robin Milner

Robin Milner: A Complete Inference System for a Class of Regular Behaviours. J. Comput. Syst. Sci. 28(3): 439-466 (1984)

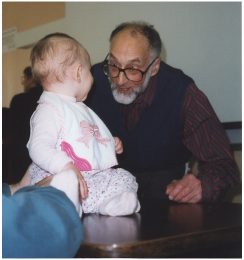
Motivation

$P ::= \mathbf{0} \mid a.P \mid P + P \mid \mu x.P^g$

Kleene-like theorem: behaviours of LTS are characterized by P's and vice-versa

Axiomatization:
$P + Q \equiv Q + P; P + \mathbf{0} \equiv P; \mu x.P \equiv P[\mu x.P/x]; \ldots$

Soundness and Completeness:
$P \equiv Q \iff P \sim Q$



Robin Milner

Robin Milner: A Complete Inference System for a Class of Regular Behaviours. J. Comput. Syst. Sci. 28(3): 439-466 (1984)

Motivation

Milner's language + axiomatization +

$a.P + a.Q \equiv a.(P + Q)$

Soundness and Completeness:
$P \equiv Q \iff tr(P) = tr(Q)$

Alexander Rabinovich

Motivation

Milner's language + axiomatization +

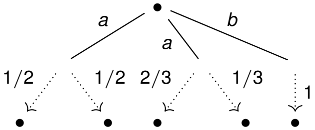
$a.P + a.Q \equiv a.(P + Q)$

Soundness and Completeness:
$P \equiv Q \iff tr(P) = tr(Q)$

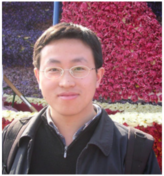
Alexander Rabinovich

Probabilistic extensions of Milner's work

Segala systems



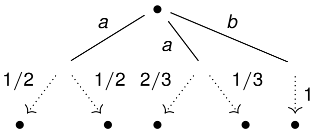
Catuscia Palamidessi



Yuxin Deng

Probabilistic extensions of Milner's work

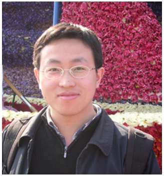
Segala systems



$$\mathsf{E} ::= \mathbf{0} \mid \mathsf{E} \boxplus \mathsf{E} \mid \mu x.\mathsf{E} \mid x \mid a \cdot \mathsf{E}'$$
$$\mathsf{E}' ::= \bigoplus_{i \in 1 \cdots n} p_i \cdot \mathsf{E}_i$$

where $a \in A$, $p_i \in (0, 1]$ and $\sum_{i \in 1 \ldots n} p_i = 1$

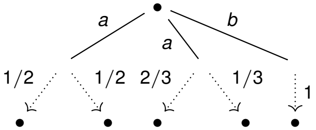
Catuscia Palamidessi



Yuxin Deng

Probabilistic extensions of Milner's work

Segala systems

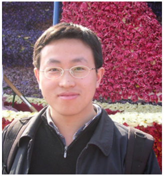


$a \cdot (1/2 \cdot \mathbf{0} \oplus 1/2 \cdot \mathbf{0}) \boxplus a \cdot (1/3 \cdot \mathbf{0} \oplus 2/3 \cdot \mathbf{0}) \boxplus b \cdot 1 \cdot \mathbf{0}$

$$\mathrm{E} ::= \mathbf{0} \mid \mathrm{E} \boxplus \mathrm{E} \mid \mu x.\mathrm{E} \mid x \mid a \cdot \mathrm{E}'$$
$$\mathrm{E}' ::= \bigoplus_{i \in 1 \cdots n} p_i \cdot \mathrm{E}_i$$

where $a \in A$, $p_i \in (0, 1]$ and $\sum_{i \in 1 \ldots n} p_i = 1$

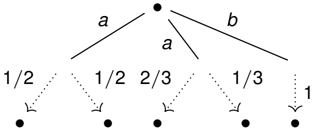
Catuscia Palamidessi



Yuxin Deng

Probabilistic extensions of Milner's work

Segala systems



$$a \cdot (1/2 \cdot \mathbf{0} \oplus 1/2 \cdot \mathbf{0}) \boxplus a \cdot (1/3 \cdot \mathbf{0} \oplus 2/3 \cdot \mathbf{0}) \boxplus b \cdot 1 \cdot \mathbf{0}$$

$$\mathrm{E} ::= \mathbf{0} \mid \mathrm{E} \boxplus \mathrm{E} \mid \mu x.\mathrm{E} \mid x \mid a \cdot \mathrm{E}'$$
$$\mathrm{E}' ::= \bigoplus_{i \in 1 \cdots n} p_i \cdot \mathrm{E}_i$$

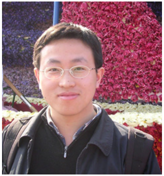
where $a \in A$, $p_i \in (0,1]$ and $\sum_{i \in 1 \ldots n} p_i = 1$

$$(\mathrm{E}_1 \boxplus \mathrm{E}_2) \boxplus \mathrm{E}_3 \equiv \mathrm{E}_1 \boxplus (\mathrm{E}_2 \boxplus \mathrm{E}_3)$$

$$\vdots$$

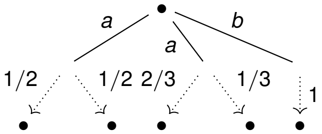
$$(p_1 \cdot \mathrm{E}) \oplus (p_2 \cdot \mathrm{E}) \equiv (p_1 + p_2) \cdot \mathrm{E}$$

Catuscia Palamidessi



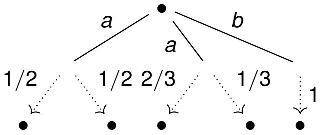
Yuxin Deng

$(p_1 \cdot \mathrm{E}) \oplus (p_2 \cdot \mathrm{E}) \equiv (p_1 + p_2) \cdot \mathrm{E}$



$$a \cdot (1/2 \cdot \mathbf{0} \oplus 1/2 \cdot \mathbf{0}) \boxplus a \cdot (1/3 \cdot \mathbf{0} \oplus 2/3 \cdot \mathbf{0}) \boxplus b \cdot 1 \cdot \mathbf{0}$$

$(p_1 \cdot \mathrm{E}) \oplus (p_2 \cdot \mathrm{E}) \equiv (p_1 + p_2) \cdot \mathrm{E}$

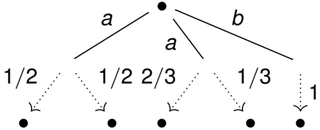


$$a \cdot (1/2 \cdot \mathbf{0} \oplus 1/2 \cdot \mathbf{0}) \boxplus a \cdot (1/3 \cdot \mathbf{0} \oplus 2/3 \cdot \mathbf{0}) \boxplus b \cdot 1 \cdot \mathbf{0}$$

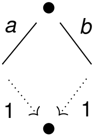
$$\equiv$$

$$a \cdot (1/2 + 1/2) \cdot \mathbf{0} \boxplus a \cdot (2/3 + 1/3) \cdot \mathbf{0} \boxplus b \cdot 1 \cdot \mathbf{0}$$

$(p_1 \cdot \mathrm{E}) \oplus (p_2 \cdot \mathrm{E}) \equiv (p_1 + p_2) \cdot \mathrm{E}$



$$a \cdot (1/2 \cdot \mathbf{0} \oplus 1/2 \cdot \mathbf{0}) \boxplus a \cdot (1/3 \cdot \mathbf{0} \oplus 2/3 \cdot \mathbf{0}) \boxplus b \cdot 1 \cdot \mathbf{0}$$
$$\equiv$$
$$a \cdot (1/2 + 1/2) \cdot \mathbf{0} \boxplus a \cdot (2/3 + 1/3) \cdot \mathbf{0} \boxplus b \cdot 1 \cdot \mathbf{0}$$
$$\equiv$$
$$a \cdot 1 \cdot \mathbf{0} \boxplus a \cdot 1 \cdot \mathbf{0} \boxplus b \cdot 1 \cdot \mathbf{0}$$



Probabilistic extensions of Milner's work II

Extensions of Milner's work uniformly to a large class of systems including Segala systems, generative systems, alternating systems, ...

Key idea: $S \to GS$

The type G is enough to derive:

1. canonical notion of equivalence (bisimilarity)
2. syntax
3. sound and complete axiomatizations

Filippo Bonchi

M. Bonsangue

Jan Rutten

Probabilistic extensions of Milner's work II

Extensions of Milner's work uniformly to a large class of systems including Segala systems, generative systems, alternating systems, ...

Key idea: $S \to GS$

The type G is enough to derive:

1. canonical notion of equivalence (bisimilarity)
2. syntax
3. sound and complete axiomatizations

Filippo Bonchi

M. Bonsangue

Jan Rutten

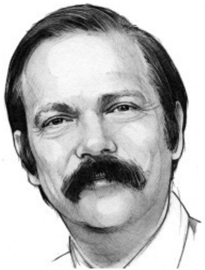
How about trace?

Many people think that bisimilarity is not the right equivalence…

…and that trace equivalence is more appropriate to reason about systems.

$$\bullet \xrightarrow{a,\frac{1}{2}} \bullet \xrightarrow{b,\frac{1}{3}} \bullet \xrightarrow{1} *$$

$$\bullet \xrightarrow{a,\frac{1}{3}} \bullet \xrightarrow{b,\frac{1}{2}} \bullet \xrightarrow{1} *$$



1. How far can Rabinovich's method be extended?
2. Can we extend sound and complete axiomatizations of probabilistic systems for bisimilarity to trace equivalence?

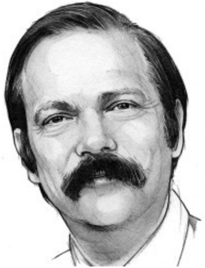
How about trace?

Many people think that bisimilarity is not the right equivalence…

…and that trace equivalence is more appropriate to reason about systems.

$$\bullet \xrightarrow{a,\frac{1}{2}} \bullet \xrightarrow{b,\frac{1}{3}} \bullet \xrightarrow{1} *$$

$$\bullet \xrightarrow{a,\frac{1}{3}} \bullet \xrightarrow{b,\frac{1}{2}} \bullet \xrightarrow{1} *$$



1. How far can Rabinovich's method be extended?
2. Can we extend sound and complete axiomatizations of probabilistic systems for bisimilarity to trace equivalence?

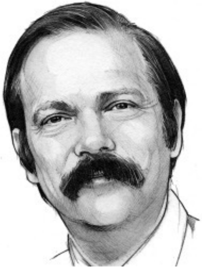
How about trace?

Many people think that bisimilarity is not the right equivalence…

…and that trace equivalence is more appropriate to reason about systems.

$$\bullet \xrightarrow{a,\frac{1}{2}} \bullet \xrightarrow{b,\frac{1}{3}} \bullet \xrightarrow{1} *$$

$$\bullet \xrightarrow{a,\frac{1}{3}} \bullet \xrightarrow{b,\frac{1}{2}} \bullet \xrightarrow{1} *$$



1. How far can Rabinovich's method be extended?
2. Can we extend sound and complete axiomatizations of probabilistic systems for bisimilarity to trace equivalence?

How about trace?

Many people think that bisimilarity is not the right equivalence. . .
. . . and that trace equivalence is more appropriate to reason about systems.

$$\bullet \xrightarrow{a,\frac{1}{2}} \bullet \xrightarrow{b,\frac{1}{3}} \bullet \xrightarrow{1} *$$

$$\bullet \xrightarrow{a,\frac{1}{3}} \bullet \xrightarrow{b,\frac{1}{2}} \bullet \xrightarrow{1} *$$

1. How far can Rabinovich's method be extended?
2. Can we extend sound and complete axiomatizations of probabilistic systems for bisimilarity to trace equivalence?

First observation: this is a general phenomenon!

Theorem (Bonsangue&Milius&Silva 2011)

Sound and complete axiomatizations for bisimilarity can always be extended to sound and complete axiomatizations for trace semantics.

The theorem is valid for a large class of systems including LTS and weighted automata, but. . . not for probabilistic systems.

This talk: the method also works for probabilistic systems!

First observation: this is a general phenomenon!

Theorem (Bonsangue&Milius&Silva 2011)

Sound and complete axiomatizations for bisimilarity can always be extended to sound and complete axiomatizations for trace semantics.

The theorem is valid for a large class of systems including LTS and weighted automata, but. . . not for probabilistic systems.

This talk: the method also works for probabilistic systems!

Generative systems

This talk will be about one type of probabilistic systems: generative systems.

$(S, \alpha \colon X \to \mathcal{D}_\omega(1 + A \times X))$

$$x \xrightarrow{p} * \quad \text{if} \quad \alpha(x)(*) = p,$$

i.e., x successfully terminates with probability p, and

$$x \xrightarrow{a,p} y \quad \text{if} \quad \alpha(x)(a, y) = p,$$

i.e., if x can make an a-labelled step to y with weight p.

Generative systems

This talk will be about one type of probabilistic systems: generative systems.

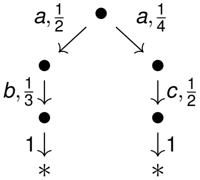
$(S, \alpha\colon\ X \to \mathcal{D}_\omega(1 + A \times X)\ )$

$$x \xrightarrow{p} * \quad \text{if} \quad \alpha(x)(*) = p,$$

i.e., x successfully terminates with probability p, and

$$x \xrightarrow{a,p} y \quad \text{if} \quad \alpha(x)(a, y) = p,$$

i.e., if x can make an a-labelled step to y with weight p.



Starting point

$$
\begin{aligned}
\mathsf{E} &::= \bigoplus_{i\in I} p_i \cdot \mathsf{F}_i \mid \mu x.\mathsf{E}^g \mid x && (p_i \in [0,1], \sum_{i\in I} p_i \le 1)\\
\mathsf{E}^g &::= \bigoplus_{i\in I} p_i \cdot \mathsf{F}_i \mid \mu x.\mathsf{E}^g && (p_i \in [0,1], \sum_{i\in I} p_i \le 1)\\
\mathsf{F}_i &::= * \mid a \cdot \mathsf{E}
\end{aligned}
$$

Bonchi et al. 2009

There is a sound and complete axiomatization w.r.t. $\sim$.

Starting point

$$
\begin{aligned}
\mathsf{E} &::= \bigoplus_{i\in I} p_i \cdot \mathsf{F}_i \mid \mu x.\mathsf{E}^g \mid x && (p_i \in [0,1], \sum_{i\in I} p_i \leq 1)\\
\mathsf{E}^g &::= \bigoplus_{i\in I} p_i \cdot \mathsf{F}_i \mid \mu x.\mathsf{E}^g && (p_i \in [0,1], \sum_{i\in I} p_i \leq 1)\\
\mathsf{F}_i &::= * \mid a \cdot \mathsf{E}
\end{aligned}
$$

Bonchi et al. 2009

There is a sound and complete axiomatization w.r.t. $\sim$.

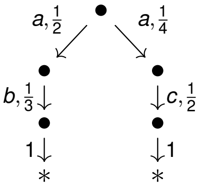
Examples

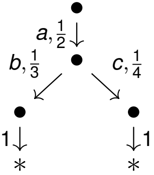
(1)
$$\begin{tikzcd}
& \bullet \arrow[dl, "{a,\frac{1}{2}}"'] \arrow[dr, "{a,\frac{1}{4}}"] & \\
\bullet \arrow[d, "{b,\frac{1}{3}}"'] & & \bullet \arrow[d, "{c,\frac{1}{2}}"] \\
\bullet \arrow[d, "1"'] & & \bullet \arrow[d, "1"] \\
* & & *
\end{tikzcd}$$

$$\frac{1}{2} \cdot a \cdot \frac{1}{3} \cdot b \cdot 1 \cdot * \oplus \frac{1}{4} \cdot a \cdot \frac{1}{2} \cdot c \cdot 1 \cdot *$$

Examples

(1)

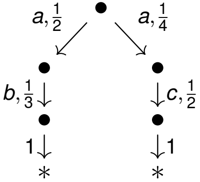


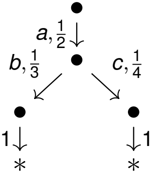


$\frac{1}{2} \cdot a \cdot \frac{1}{3} \cdot b \cdot 1 \cdot * \oplus \frac{1}{4} \cdot a \cdot \frac{1}{2} \cdot c \cdot 1 \cdot *$ $\quad$ $\frac{1}{2} \cdot a \cdot \left(\frac{1}{3} \cdot b \cdot 1 \cdot * \oplus \frac{1}{4} \cdot c \cdot 1 \cdot *\right)$

Examples

(1)





$\frac{1}{2} \cdot a \cdot \frac{1}{3} \cdot b \cdot 1 \cdot * \oplus \frac{1}{4} \cdot a \cdot \frac{1}{2} \cdot c \cdot 1 \cdot *$ $\quad$ $\frac{1}{2} \cdot a \cdot \left(\frac{1}{3} \cdot b \cdot 1 \cdot * \oplus \frac{1}{4} \cdot c \cdot 1 \cdot *\right)$

Top states are not bisimilar but they are trace equivalent.

$$ab \mapsto \frac{1}{6}; \qquad ab \mapsto \frac{1}{8}$$

Axiomatization

$$\vdots$$

all the axioms for $\sim$

$$\vdots$$

$$p \cdot a \cdot (p_1 \mathrm{E}_1 \oplus p_2 \mathrm{E}_2) \quad \equiv \quad p_1 \cdot a \cdot p\mathrm{E}_1 \oplus p_2 \cdot a \cdot p\mathrm{E}_2 \quad (D)$$

Part of (D) is about multiplying probabilities

We define a notion of *scalar product* for expressions:

$$p\left(\bigoplus_{i\in I} p_i \cdot \mathsf{F}_i\right) = \bigoplus_{i\in I}(pp_i)\cdot \mathsf{F}_i$$

$$\bullet \xrightarrow{a,\frac{1}{2}} \bullet \xrightarrow{b,\frac{1}{3}} \bullet \xrightarrow{1} *$$

$$\bullet \xrightarrow{a,\frac{1}{3}} \bullet \xrightarrow{b,\frac{1}{2}} \bullet \xrightarrow{1} *$$

$$\frac{1}{2}\cdot a\cdot \frac{1}{3}\cdot b\cdot 1\cdot * \equiv \frac{1}{3}\cdot\frac{3}{2}(a\cdot\frac{1}{3}\cdot b\cdot 1\cdot *) = \frac{1}{3}\cdot a\cdot\frac{1}{2}\cdot b\cdot 1\cdot *$$

Part of (D) is about multiplying probabilities

We define a notion of *scalar product* for expressions:

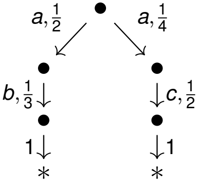
$$p\left(\bigoplus_{i\in I} p_i \cdot \mathsf{F}_i\right) = \bigoplus_{i\in I} (pp_i)\cdot \mathsf{F}_i$$

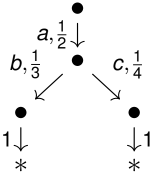
$$\bullet \xrightarrow{a,\frac{1}{2}} \bullet \xrightarrow{b,\frac{1}{3}} \bullet \xrightarrow{1} *$$

$$\bullet \xrightarrow{a,\frac{1}{3}} \bullet \xrightarrow{b,\frac{1}{2}} \bullet \xrightarrow{1} *$$

$$\frac{1}{2}\cdot a\cdot \frac{1}{3}\cdot b\cdot 1\cdot * \equiv \frac{1}{3}\cdot\frac{3}{2}(a\cdot\frac{1}{3}\cdot b\cdot 1\cdot *) = \frac{1}{3}\cdot a\cdot\frac{1}{2}\cdot b\cdot 1\cdot *$$

(D) is also about eliminating branching

(1) 



$$\left(\tfrac{1}{2} \cdot a \cdot \tfrac{1}{3} \cdot b \cdot 1 \cdot *\right) \oplus \left(\tfrac{1}{4} \cdot a \cdot \tfrac{1}{2} \cdot c \cdot 1 \cdot *\right) \qquad \left(\tfrac{1}{2} \cdot a \cdot \left(\tfrac{1}{3} \cdot b \cdot 1 \cdot *\right) \oplus \tfrac{1}{4} \cdot c \cdot 1 \cdot\right.$$

$$\overset{(D)}{\equiv} \qquad\qquad =$$

$$\tfrac{1}{2} \cdot a \cdot \left(\tfrac{1}{2} \left(\tfrac{2}{3} \cdot b \cdot 1 \cdot *\right) \oplus \tfrac{1}{4}(1 \cdot c \cdot 1 \cdot *\right)$$

Soundness and Completeness

- Soundness and completeness proofs often boil down to find *normal forms*;
- Rabinovich's proof uses the fact that every finite LTS can be changed to a finite trace-equivalent LTS that is deterministic.
- This is not so trivial for probabilistic systems: for a finite system, there may be no finite deterministic system that is trace equivalent to it.
- We will use an (infinite) determinization of a probabilistic transition system but avoid reasoning about normal forms by using a coinductive approach.

Soundness and Completeness

- Soundness and completeness proofs often boil down to find *normal forms*;
- Rabinovich's proof uses the fact that every finite LTS can be changed to a finite trace-equivalent LTS that is deterministic.
- This is not so trivial for probabilistic systems: for a finite system, there may be no finite deterministic system that is trace equivalent to it.
- We will use an (infinite) determinization of a probabilistic transition system but avoid reasoning about normal forms by using a coinductive approach.

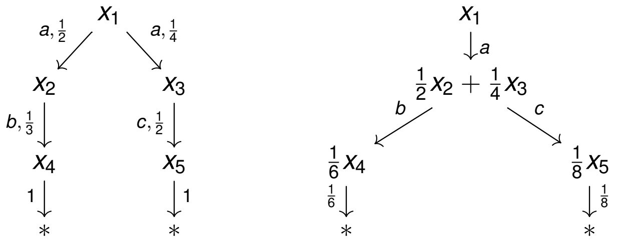
Soundness and Completeness

- Soundness and completeness proofs often boil down to find *normal forms*;
- Rabinovich's proof uses the fact that every finite LTS can be changed to a finite trace-equivalent LTS that is deterministic.
- This is not so trivial for probabilistic systems: for a finite system, there may be no finite deterministic system that is trace equivalent to it.
- We will use an (infinite) determinization of a probabilistic transition system but avoid reasoning about normal forms by using a coinductive approach.

General strategy

- Determinization

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, "\alpha"'] & \mathcal{D}_\omega(X) \arrow[ddl, "(\delta\circ\alpha)^\sharp"] \\
\mathcal{D}_\omega(1 + A \times X) \arrow[d, "\delta"'] & \\
{[0,1] \times \mathcal{D}_\omega(X)^A} &
\end{tikzcd}$$



- Determinization and semantics by finality

$$
\begin{tikzcd}
X \arrow[rr, "\eta_X"] \arrow[d, "\alpha"'] & & \mathcal{D}_\omega(X) \arrow[r, dashed, "out"] \arrow[dll, "(\delta\circ\alpha)^\sharp"] & {[0,1]^{A^*}} \arrow[dd, "{\langle \varepsilon ?, (-)_a \rangle}"] \\
\mathcal{D}_\omega(1 + A \times X) \arrow[d, "\delta"'] & & & \\
{[0,1] \times \mathcal{D}_\omega(X)^A} \arrow[rrr, dashed, "id \times out^A"] & & & {[0,1] \times ([0,1]^{A^*})^A}
\end{tikzcd}
$$

General strategy

- Determinization and semantics by finality

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, "\alpha"'] & \mathcal{D}_\omega(X) \arrow[r, dashed, "out"] \arrow[ddl, "(\delta\circ\alpha)^\sharp"] & {[0,1]^{A^*}} \arrow[dd, "{\langle \varepsilon?, (-)_a \rangle}"] \\
\mathcal{D}_\omega(1 + A \times X) \arrow[d, "\delta"'] & & \\
{[0,1] \times \mathcal{D}_\omega(X)^A} \arrow[rr, dashed, "id \times out^A"] & & {[0,1] \times ([0,1]^{A^*})^A}
\end{tikzcd}$$

Theorem

For any $x \in X$, $tr(x) = out(\eta(x))$. For $E \in$ Exp, $tr(x) = out_{\equiv}([E])$

This actually means that the image of *out* is a distribution on words.

Soundness and completeness

Soundness	Completeness
$\mathsf{E}_1 \equiv \mathsf{E}_2$	$\mathsf{E}_1 \sim_{\mathsf{tr}} \mathsf{E}_2$
$\Leftrightarrow \quad [\mathsf{E}_1] = [\mathsf{E}_2]$	$\Leftrightarrow \quad \mathsf{tr}(\mathsf{E}_1) = tr(\mathsf{E}_2)$
$\overset{(*)}{\Rightarrow} \quad out_{\equiv}([\mathsf{E}_1]) = out_{\equiv}([\mathsf{E}_2])$	$\overset{(\triangle)}{\Leftrightarrow} \quad out_{\equiv}([\mathsf{E}_1]) = out_{\equiv}([\mathsf{E}_2])$
$\overset{(\triangle)}{\Leftrightarrow} \quad tr(\mathsf{E}_1) = tr(\mathsf{E}_2)$	$\overset{(\heartsuit)}{\Rightarrow} \quad [\mathsf{E}_1] = [\mathsf{E}_2]$
$\Leftrightarrow \quad \mathsf{E}_1 \sim_{\mathsf{tr}} \mathsf{E}_2$	$\Leftrightarrow \quad \mathsf{E}_1 \equiv \mathsf{E}_2$

$(*)$: existence of $out_{\equiv}$, $(\triangle)$: $out_{\equiv} \circ [-] = tr$, $(\heartsuit)$: $out_{\equiv}$ is injective.

The proof of $(\heartsuit)$ is where the difficulties arose.

Soundness and completeness

Soundness	Completeness
$\mathsf{E}_1 \equiv \mathsf{E}_2$	$\mathsf{E}_1 \sim_{\mathrm{tr}} \mathsf{E}_2$
$\Leftrightarrow \quad [\mathsf{E}_1] = [\mathsf{E}_2]$	$\Leftrightarrow \quad \mathrm{tr}(\mathsf{E}_1) = tr(\mathsf{E}_2)$
$\overset{(*)}{\Rightarrow} \quad out_{\equiv}([\mathsf{E}_1]) = out_{\equiv}([\mathsf{E}_2])$	$\overset{(\triangle)}{\Leftrightarrow} \quad out_{\equiv}([\mathsf{E}_1]) = out_{\equiv}([\mathsf{E}_2])$
$\overset{(\triangle)}{\Leftrightarrow} \quad tr(\mathsf{E}_1) = tr(\mathsf{E}_2)$	$\overset{(\heartsuit)}{\Rightarrow} \quad [\mathsf{E}_1] = [\mathsf{E}_2]$
$\Leftrightarrow \quad \mathsf{E}_1 \sim_{\mathrm{tr}} \mathsf{E}_2$	$\Leftrightarrow \quad \mathsf{E}_1 \equiv \mathsf{E}_2$

$(*)$: existence of $out_{\equiv}$, $(\triangle)$: $out_{\equiv} \circ [-] = tr$, $(\heartsuit)$: $out_{\equiv}$ is injective.

The proof of $(\heartsuit)$ is where the difficulties arose.

Conclusions and Future work

Conclusions

- First sound and complete axiomatization of trace semantics of generative systems
- Similarly to Rabinovich

Future Work

- Extend uniformly to other types of systems
- All the proofs are coinductive parametrized by the functor type. The restriction on generalizing lies in the the theory of generic coalgebraic trace semantics (Hasuo &Jacobs&Sokolova 2007).

Thank you for your attention!