A decision procedure for bisimilarity of generalized regular expressions

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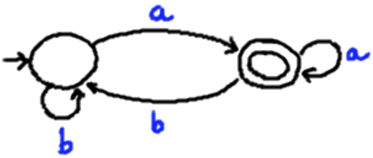
SBMF'10, November 2010

Motivation

Deterministic automata (DA)

- Widely used model in Computer Science.
- Acceptors of languages



Regular expressions

- *User-friendly* alternative to DA notation.
- Many applications: pattern matching (`grep`), specification of circuits, ...

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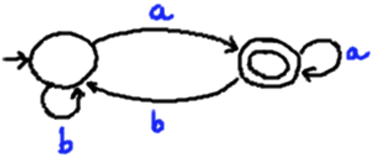

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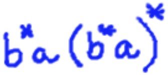
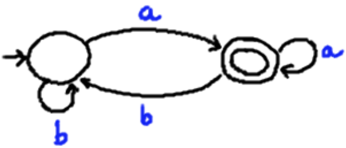


Kleene's Theorem

Let $A \subseteq \Sigma^*$. The following are equivalent.

1. $A = L(\mathcal{A})$, for some finite automaton $\mathcal{A}$.
2. $A = L(r)$, for some regular expression r.

Motivation



Kleene's theorem

$b^*a(b^*a)^*$

KA

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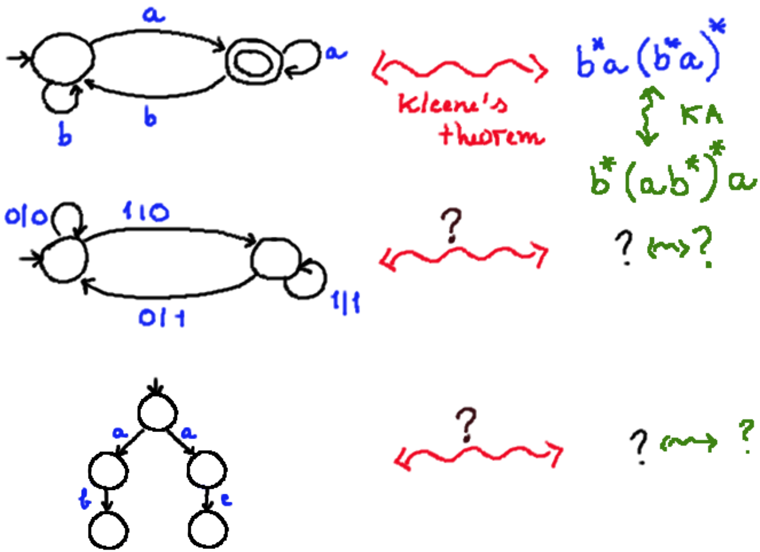

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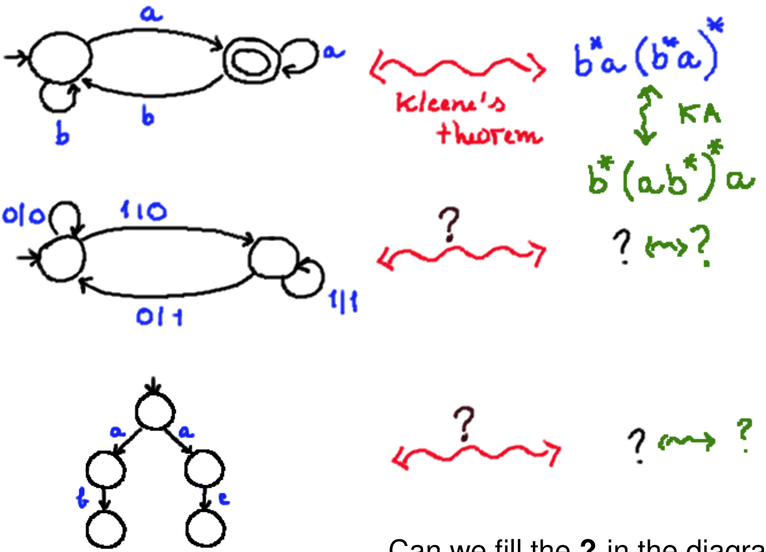

Can we fill the **?** in the diagram?

In previous work . . .

We presented:

- a generalized notion of regular expressions;
- an analogue of Kleene's theorem;
- and sound and complete axiomatizations with respect to bisimilarity

for a large class of systems (labelled transition systems, Mealy machines, probabilistic automata).

All the above was derived modularly from the type of each system.

Question: Can we automate the reasoning on equivalence of expressions, also in a modular way?

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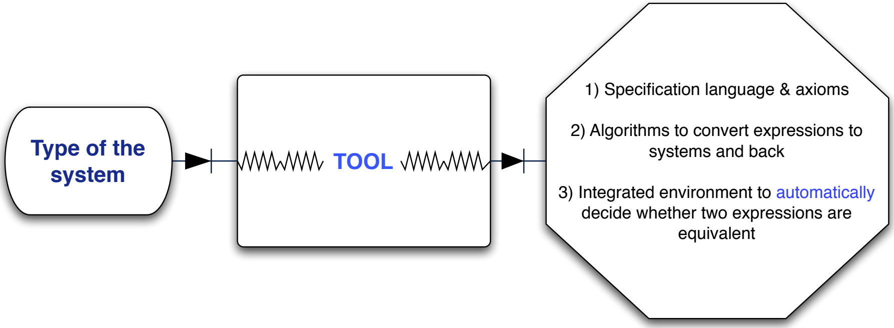
The ultimate goal. . .



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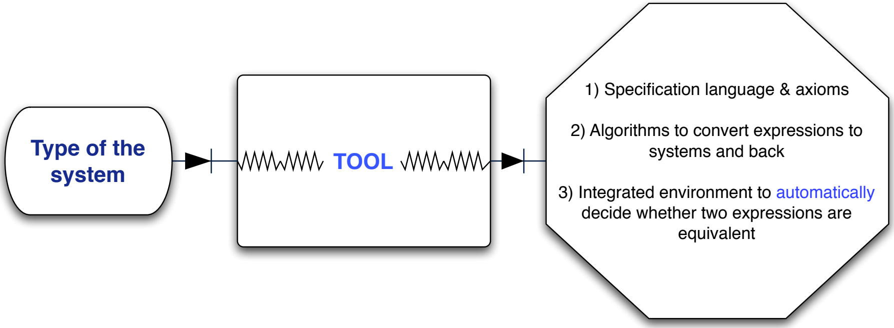
The ultimate goal...



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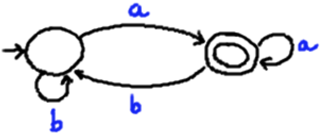
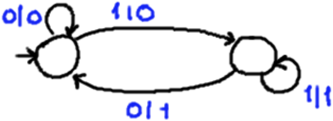
Outline

- Generalized regular expressions
- Equivalence of expressions
- Snapshot of the tool

What do these things have in common?



$(S, \delta : S \to 2 \times S^A)$



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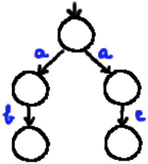

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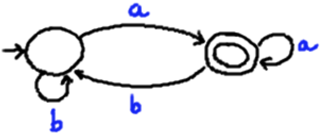

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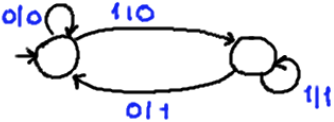

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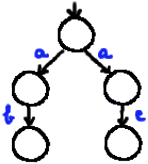

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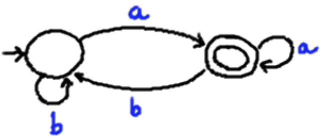

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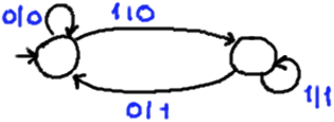

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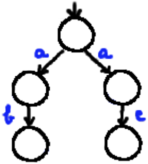

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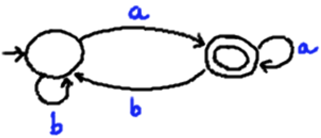

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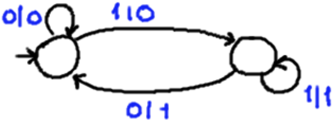

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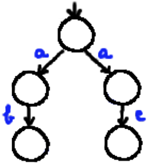

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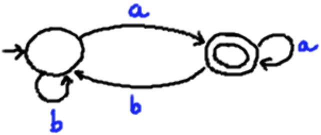

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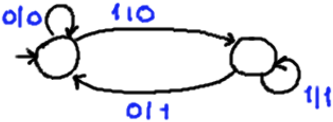

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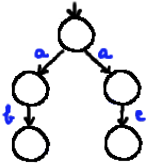

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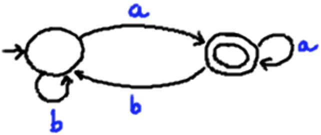

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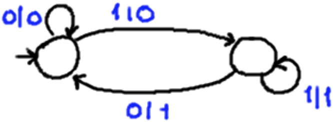

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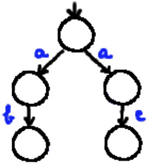

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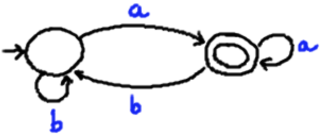

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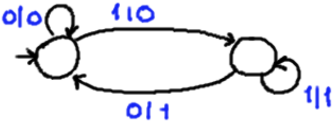

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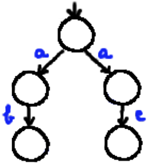

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$(S, \delta : S \to \mathcal{G}S)$ $\mathcal{G}$**-coalgebras**

Coalgebras

- Generalizations of deterministic automata
- Set of states S and a transition function $t : S \to \mathcal{G}S$ where $\mathcal{G}$ encodes the type of the system:

$$\mathcal{G} ::= Id \mid B \mid \mathcal{G} \times \mathcal{G} \mid \mathcal{G} + \mathcal{G} \mid \mathcal{G}^A \mid \mathcal{P}\mathcal{G}$$

$\mathcal{P}$ finite

Examples

- $\mathcal{G} = 2 \times Id^A$ Deterministic automata
- $\mathcal{G} = (B \times Id)^A$ Mealy machines
- $\mathcal{G} = 1 + (\mathcal{P}Id)^A$ LTS (with explicit termination)
- ...

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The power of $\mathcal{G}$

The functor $\mathcal{G}$ determines:

1. notion of observational equivalence (coalg. bisimulation)
2. behaviour (final coalgebra)
3. set of expressions describing finite systems
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1 + 2 are standard universal coalgebra; 3 + 4 are [BRS10]

In a nutshell — beyond deterministic automata

Deterministic automata $Q \to 2 \times Q^{\Sigma}$	$\rightsquigarrow$	$\mathcal{G}$-coalgebras $Q \to \mathcal{G}Q$
$\Updownarrow$		$\Updownarrow$
Regular Expressions	$\rightsquigarrow$	$\mathcal{G}$-expressions
$\Updownarrow$		$\Updownarrow$
Formal Languages	$\rightsquigarrow$	Final coalgebra

$\mathcal{G}$-expressions

$$E \quad ::= \quad \underline{\emptyset} \mid \epsilon \mid E \cdot E \mid E + E \mid E^*$$

$$E_{\mathcal{G}} \quad ::= \quad ?$$

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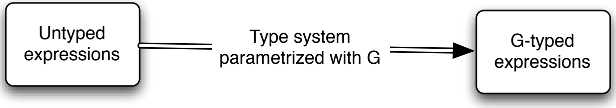
How do we define $E_{\mathcal{G}}$?



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Deterministic automata expressions – $\mathcal{G} = 2 \times Id^A$

$$\varepsilon \quad ::= \quad \underbrace{\underline{\emptyset} \mid \varepsilon \oplus \varepsilon \mid \mu x.\gamma}_{\mathcal{G}} \mid$$

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Operational semantics

The set of $\mathcal{G}$-expressions has a coalgebraic structure given by

$$\delta_{\mathcal{G}} : \mathsf{Exp}_{\mathcal{G}} \to \mathcal{G}(\mathsf{Exp}_{\mathcal{G}})$$

$\delta_{\mathcal{G}}$ ...

- ...provides an operational semantics for the set of expressions
- ...defines the dynamics of the system
- ...is used for observing the behaviour of the system

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The goal: proving equivalence

- Automatically proving equivalence (bisimilarity) of expressions;
- Tool: `Circ`

- Meta-language implemented in `Maude`
- **Input**: Algebraic specification + coalgebraic structure (dynamics)
- **Engine**: circular coinduction – constructing bisimulation
- **Output**: Yes / No / **?**

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Our challenge

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\begin{tikzcd}
\mathrm{Exp}_{\mathcal{G}} \arrow[dr, Rightarrow] & & \\
\text{coalgebraic structure}(\delta_{\mathcal{G}}) \arrow[r, Rightarrow] & \text{algebraic specification} \arrow[r] & \mathtt{CIRC} \arrow[dd, Rightarrow, "\varepsilon_1 \sim_{\mathcal{G}} \varepsilon_2"] \\
\mathcal{G} - \text{bisimilarity} \arrow[r, Rightarrow] & \text{deduction relation} \arrow[ur] & \\
 & & \textit{YES}/\textit{NO}
\end{tikzcd}
$$

How to guarantee that the prover always says YES/NO?

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Unraveling can be infinite

$$\varepsilon = \mu x.a(b(x)) \oplus 1$$

$$\varepsilon \xrightarrow{\delta_a} \langle 1, b(\varepsilon) \rangle \xrightarrow{\delta_b} \langle 1, \varepsilon \rangle$$

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\varepsilon \arrow[r, "\delta_a"] \arrow[dr, "\delta_b"'] & \langle 1, b(\varepsilon) \rangle \arrow[r, "\delta_b"] \arrow[d, "\delta_a"] & \langle 1, \varepsilon \rangle \\
& \langle 0, \emptyset \rangle &
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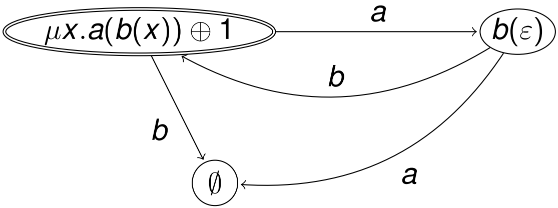

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$\varepsilon \overset{\delta}{\longmapsto} \langle 0, \varepsilon \oplus \varepsilon \rangle$

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We need **ACI**!

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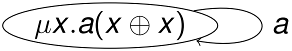
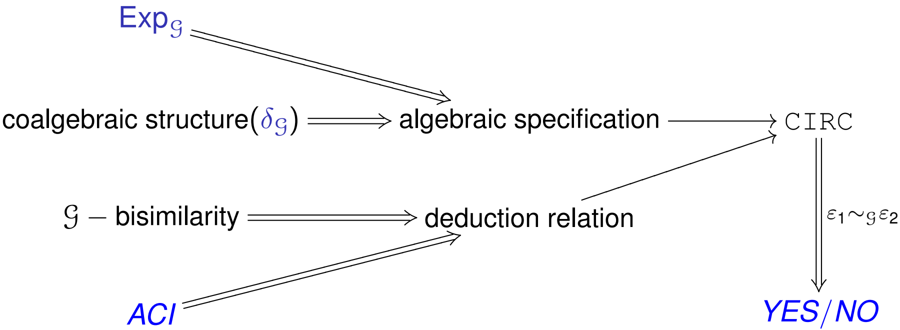


A decision procedure for bisimilarity



- Adding equations is possible because `Circ` is an extension of `Maude`
- With ACI, the *bisimulation game* is decidable!

Algebra meets coalgebra

A decision procedure for bisimilarity

$$\begin{tikzcd}
\mathrm{Exp}_{\mathcal{G}} \arrow[dr, Rightarrow] & & \\
\text{coalgebraic structure}(\delta_{\mathcal{G}}) \arrow[r, Rightarrow] & \text{algebraic specification} \arrow[r] & \mathtt{CIRC} \arrow[dd, Rightarrow, "\varepsilon_1 \sim_{\mathcal{G}} \varepsilon_2"] \\
\mathcal{G} - \text{bisimilarity} \arrow[r, Rightarrow] & \text{deduction relation} \arrow[ur] & \\
\mathit{ACI} \arrow[ur, Rightarrow] & & \mathit{YES}/\mathit{NO}
\end{tikzcd}$$

- Adding equations is possible because `Circ` is an extension of `Maude`
- With ACI, the *bisimulation game* is decidable!

Algebra meets coalgebra

Example

$$
\begin{aligned}
\varepsilon_X &= \mu x.a.x \oplus b.\underline{\emptyset} \oplus a.(\mu y.a.y \oplus b.\underline{\emptyset} \oplus b.b.\surd) \oplus b.b.\surd \\
\varepsilon_Z &= \mu z.a.z \oplus b.\underline{\emptyset} \oplus b.b.\surd
\end{aligned}
$$

Question Are ε_X and ε_Z equivalent?

Example

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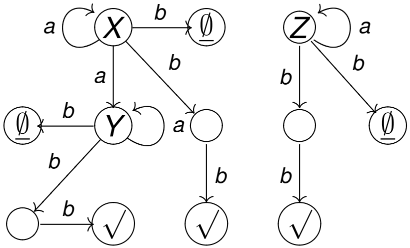

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Tool Snapshot

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Maude> in protofunctorizer.maude
(alph A is a b endalph)
(alph B is tick endalph)
(functor B + P(Id)^A .)
(E1 = mu X. ... )
(E2 = mu Z. ... )
(set goal E1 = E2 .)
(generate coalgebra .)
296 lines of CIRC specification!
Maude> in circ.maude XandZ.maude
(coinduction .)
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Proof succeeded.

$$\mathcal{G} = B + \mathcal{P}(Id)^A, A = \{a, b\}, B = \{\surd\}$$

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Conclusions and Future work

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- Generic framework to uniformly derive language and axioms for a large class of systems
- Generalization of Kleene theorem and Kleene algebra, parametric on the functor.
- Automation in `Circ`: decision procedure for equivalence of expressions.

Future work

- Making the tool more user friendly;
- Extending the class of systems to include quantitative systems (probabilistic automata, etc)
- Apply it to a serious case study (circuit design, compiler optimization, . . .)

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Thank you!

- for more details see:
  - “Non-deterministic Kleene coalgebras"
    A.Silva, M. Bonsangue, J. Rutten
  - “Circular coinduction - A proof theoretical foundation"
    G. Roşu, D. Lucanu
- QUESTIONS?