Coalgebraic Up-to Techniques

Alexandra Silva

(joint with Bonsangue, Bonchi, Pous, Rot & Rutten)

Radboud University Nijmegen & CWI Amsterdam

Shonan Meeting 026
07.10.2013

(slide credits: Damien Pous)

Context

Tools and proof techniques for systems equivalence

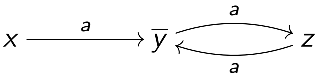
Methodology:

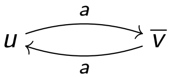
1. characterise coinductively a given notion of equivalence
2. improve the associated proof method

up-to techniques

Deterministic finite automata

The states x and u are language equivalent





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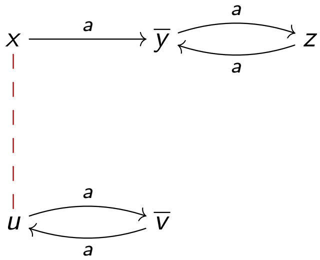

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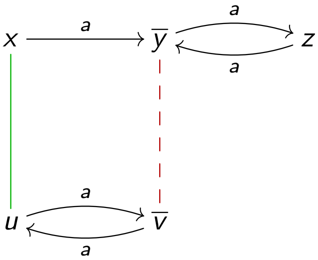

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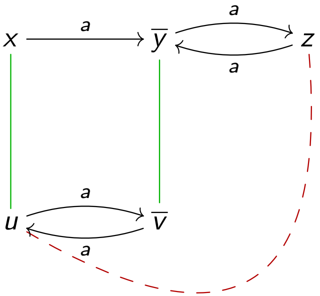

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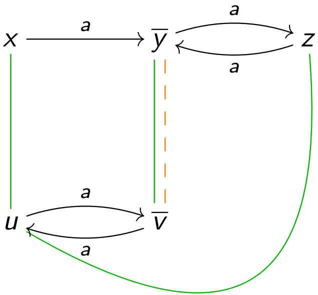

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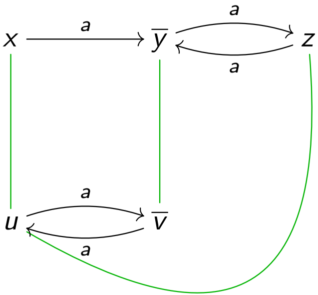

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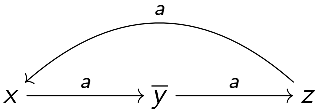

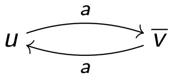

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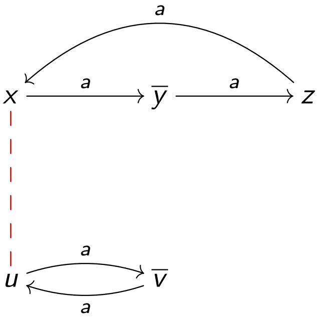

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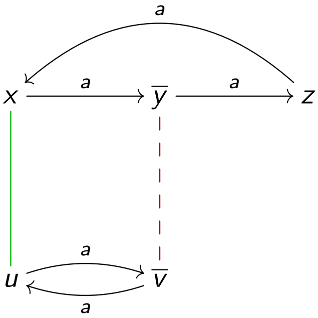

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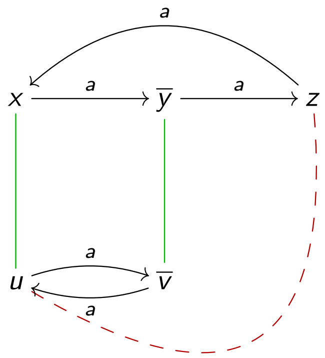

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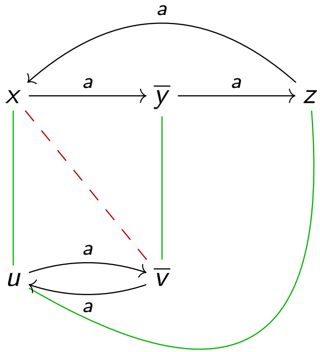

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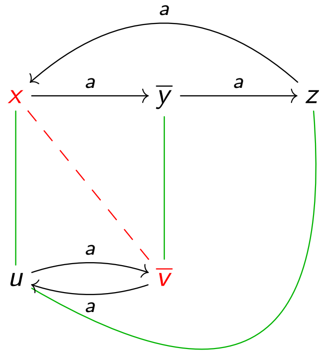

Correctness

- A relation R is a bisimulation if $x \; R \; y$ entails
  - $o(x) = o(y)$;
  - for all a, $t_a(x) \; R \; t_a(y)$.

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- *Theorem:* $L(x) = L(y)$ iff
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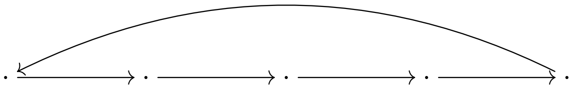
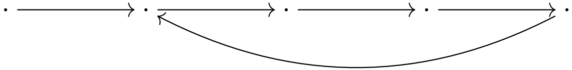
The previous algorithm attempts to construct a bisimulation

Complexity

The previous algorithm is quadratic





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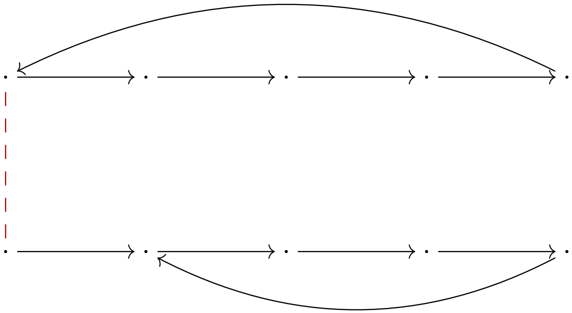

0 pairs

Complexity

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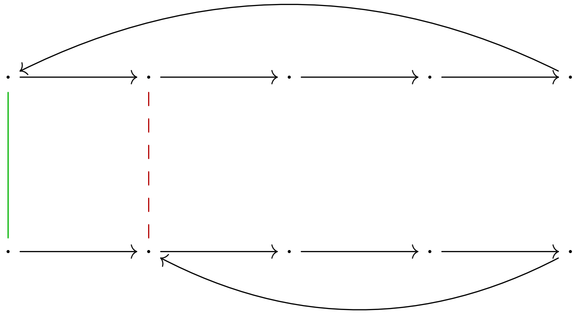

1 pairs

Complexity

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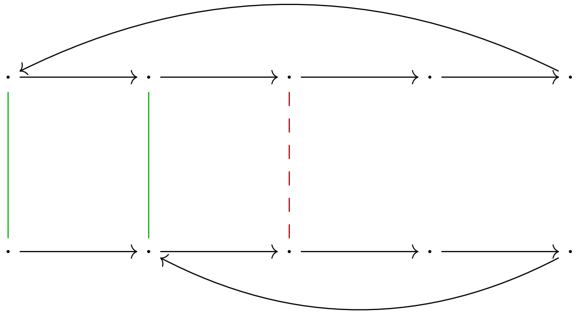

2 pairs

Complexity

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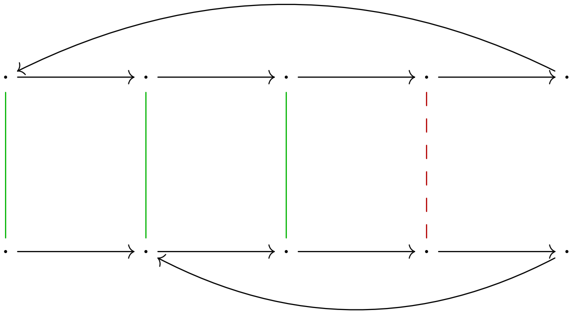

3 pairs

Complexity

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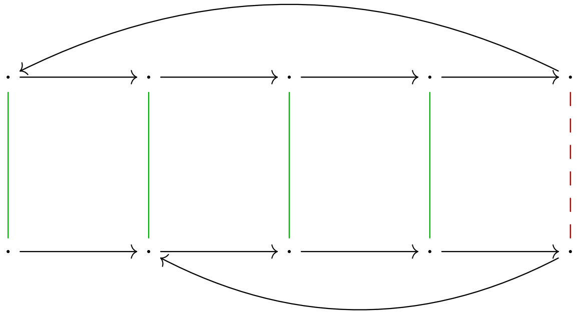

4 pairs

Complexity

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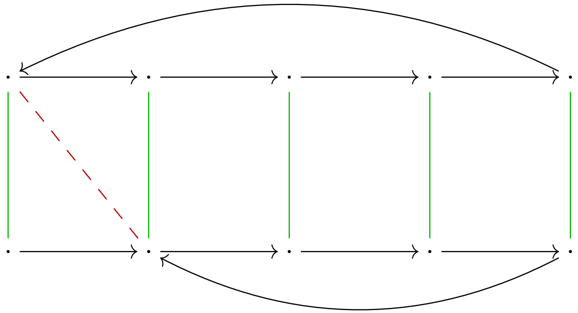

5 pairs

Complexity

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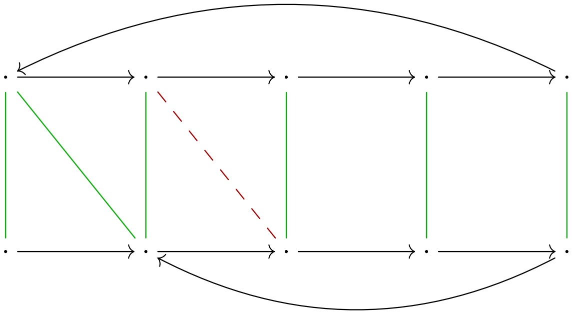

6 pairs

Complexity

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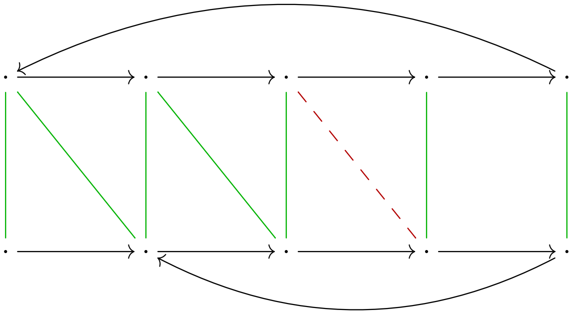

7 pairs

Complexity

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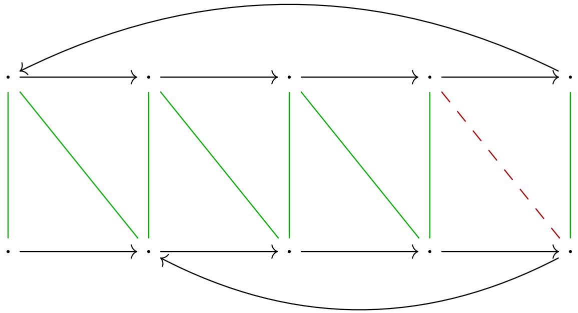

8 pairs

Complexity

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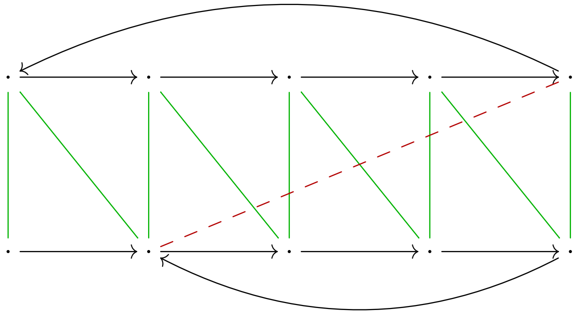
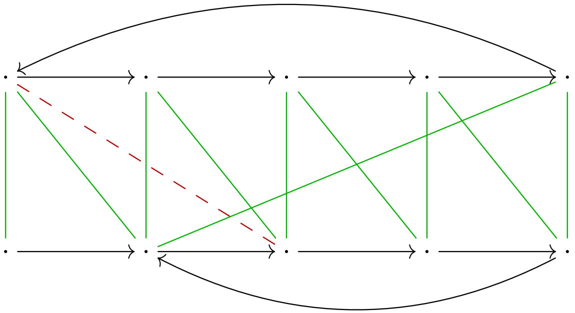


9 pairs

Complexity

The previous algorithm is quadratic



10 pairs

Complexity

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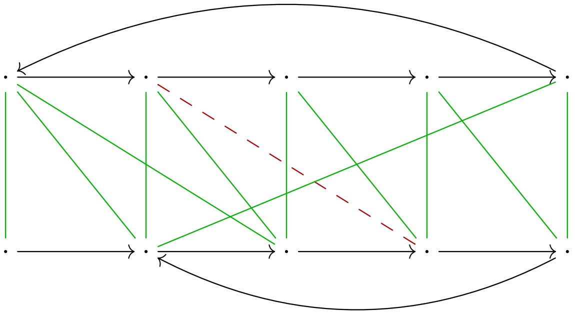

11 pairs

Complexity

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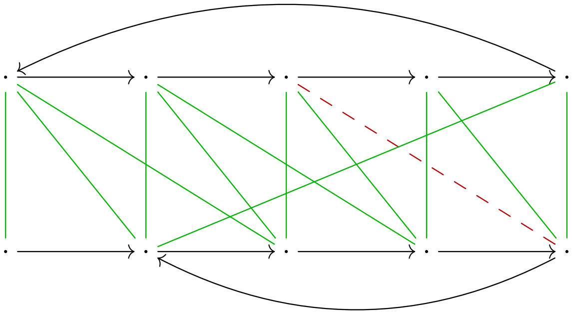

12 pairs

Complexity

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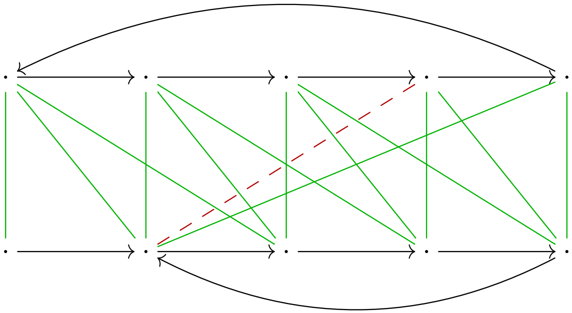

13 pairs

Complexity

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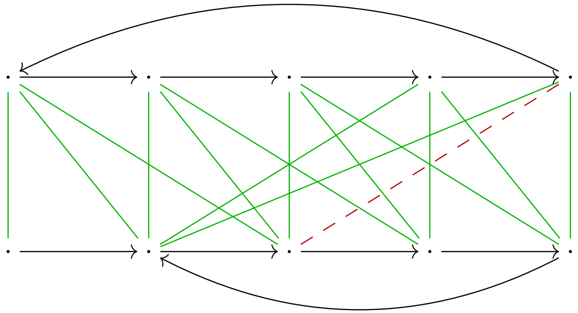

14 pairs

Complexity

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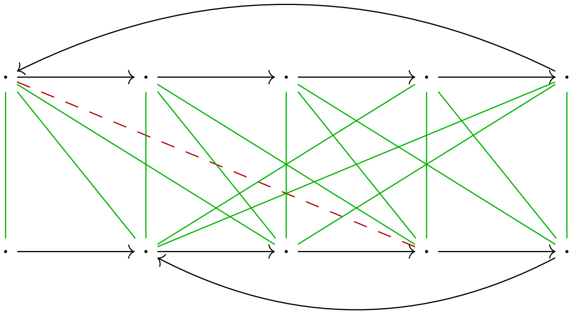

15 pairs

Complexity

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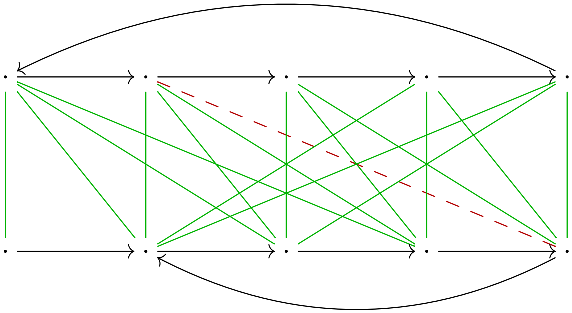

16 pairs

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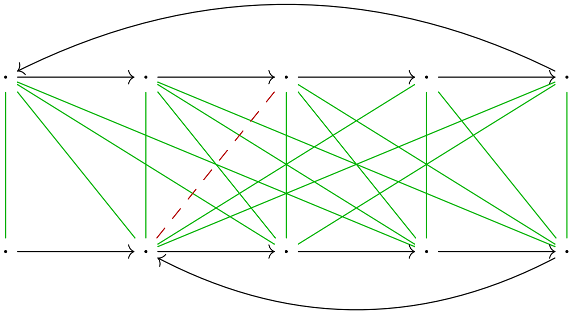

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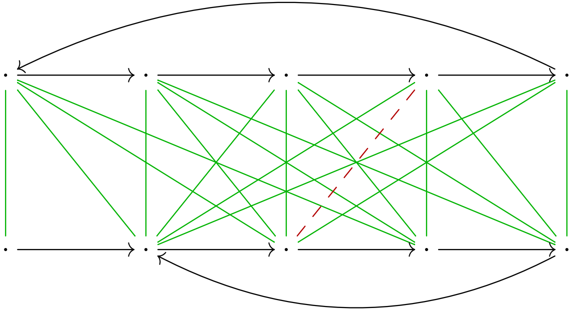

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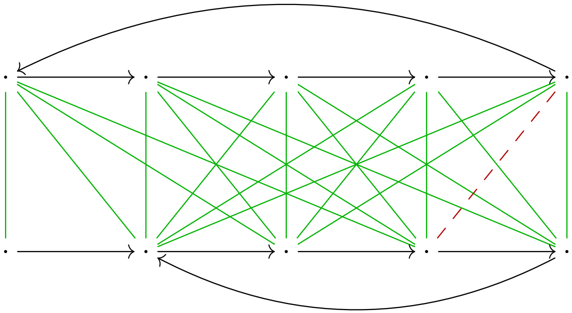

19 pairs

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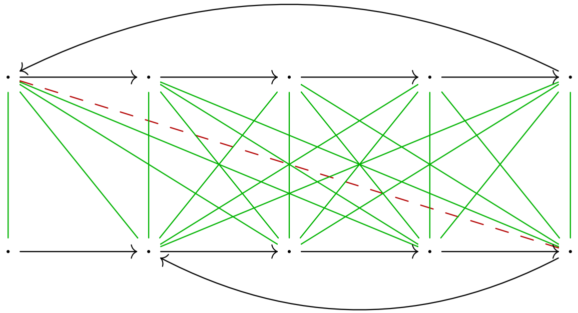

20 pairs

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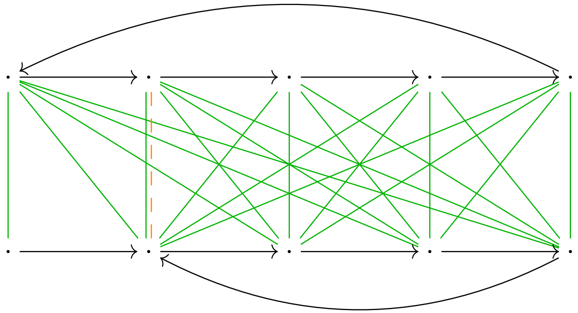

21 pairs

Complexity

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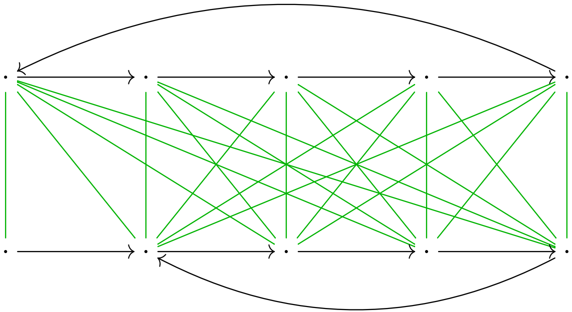
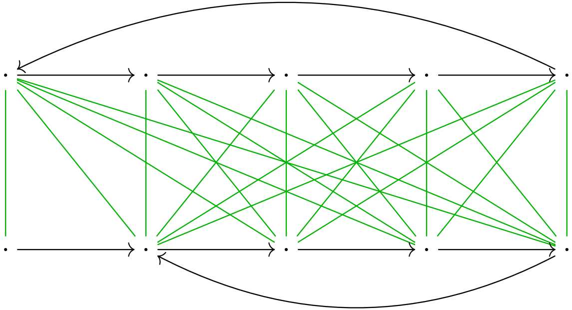


21 pairs

First improvement

One can stop much earlier



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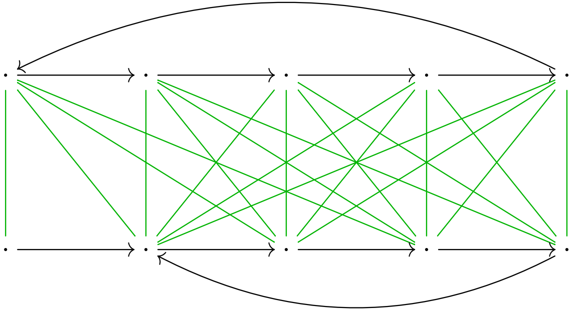

~~21~~ 20 pairs

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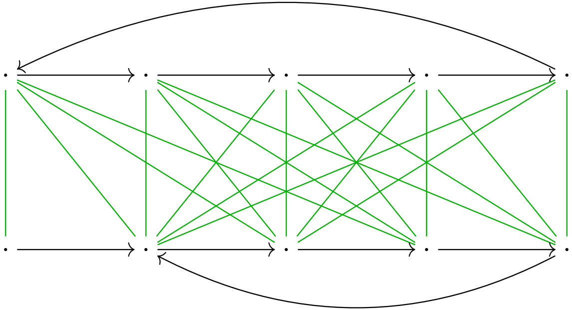

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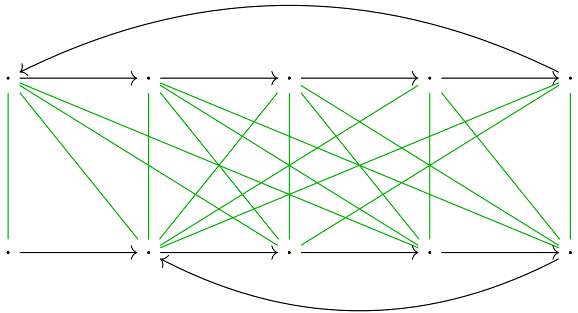

~~21~~ 18 pairs

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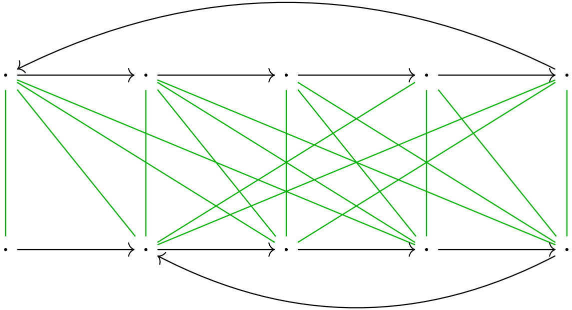

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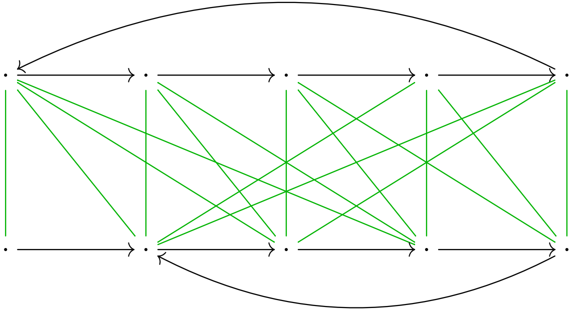

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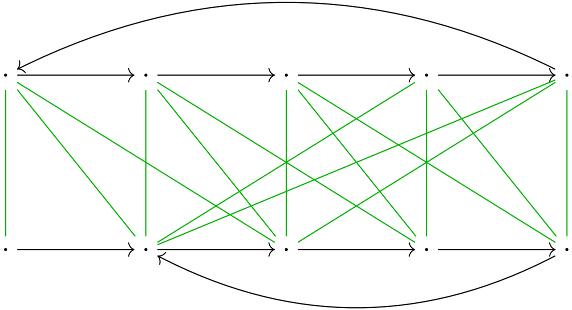

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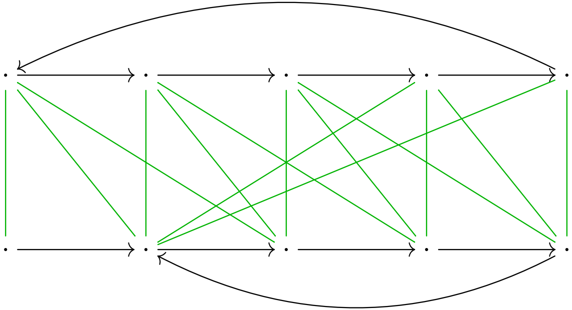

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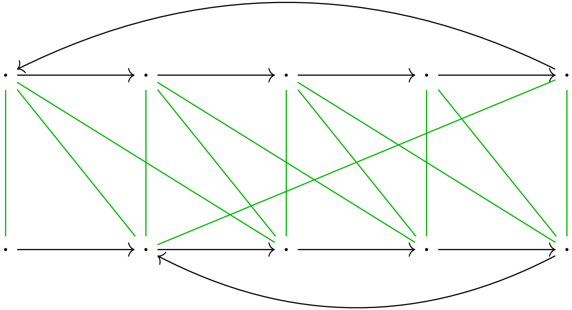

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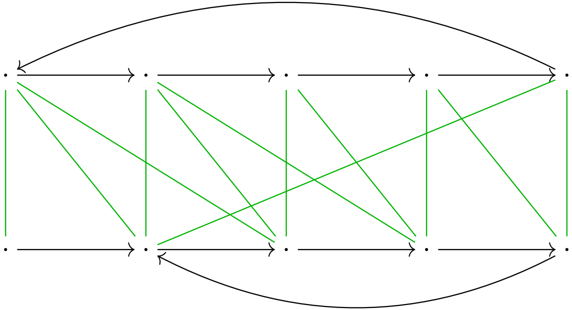

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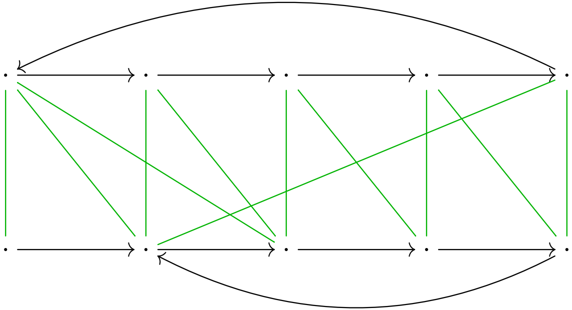

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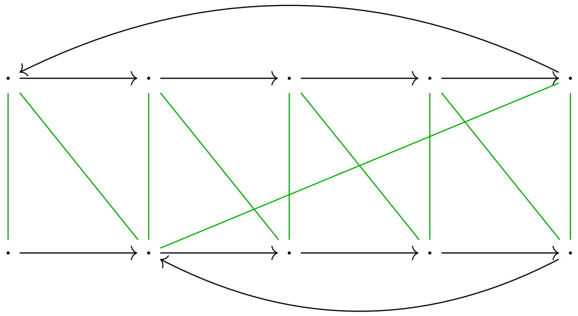

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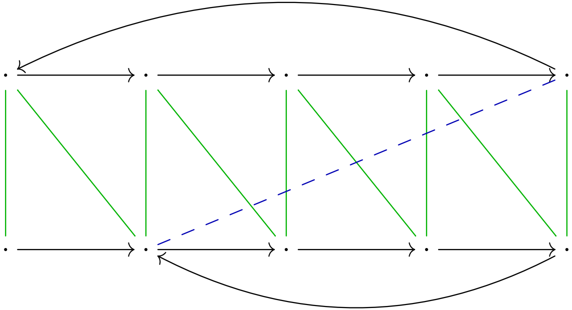

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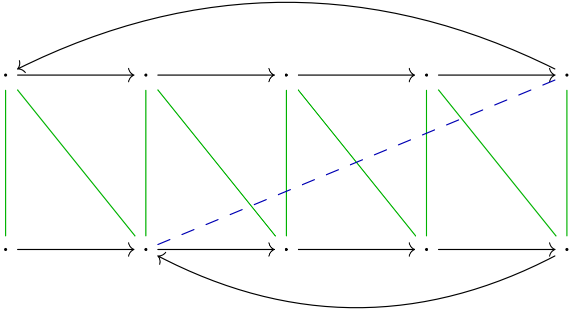

[Hopcroft and Karp '71]

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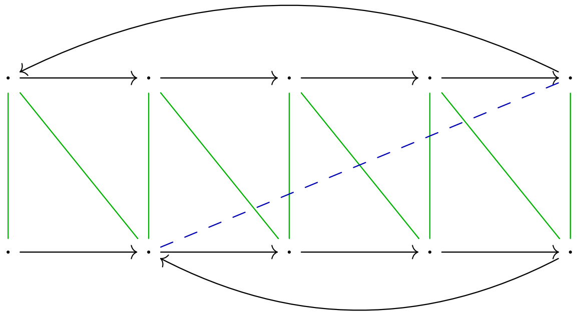

[Hopcroft and Karp '71]

Complexity: almost linear

[Tarjan '75]

Correctness of the improvement

Correctness of HK algorithm, revisited:

- Denote by R^e the equivalence closure of R
- R is a bisimulation up to equivalence if $x \; R \; y$ entails
  - $o(x) = o(y)$;
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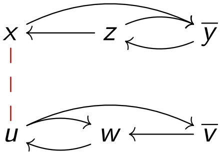
Ten years before Milner and Park!

Non-Deterministic Automata

Use Hopcroft and Karp on the fly, through the powerset construction:



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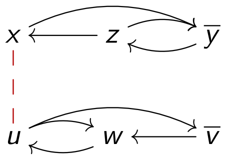

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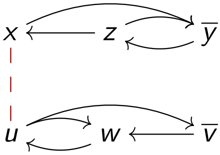

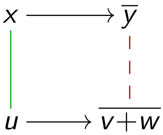

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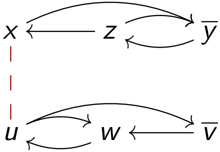

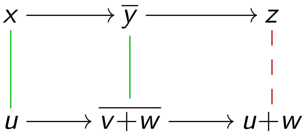

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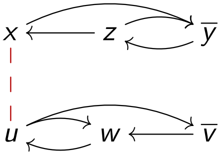

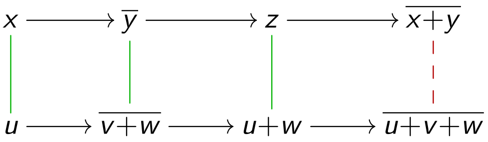

Non-Deterministic Automata

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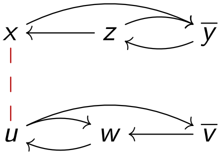

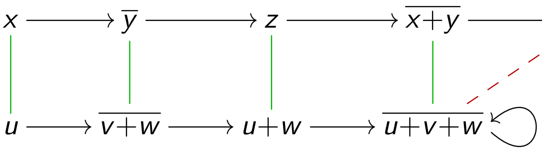

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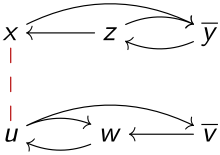

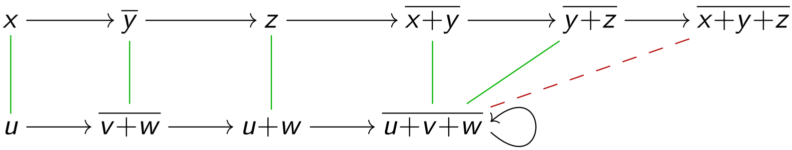

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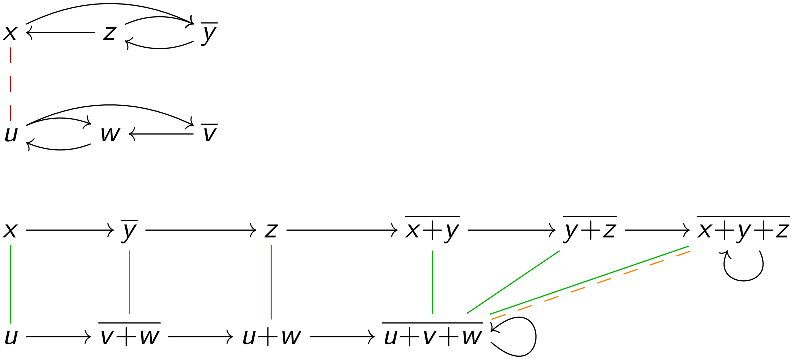

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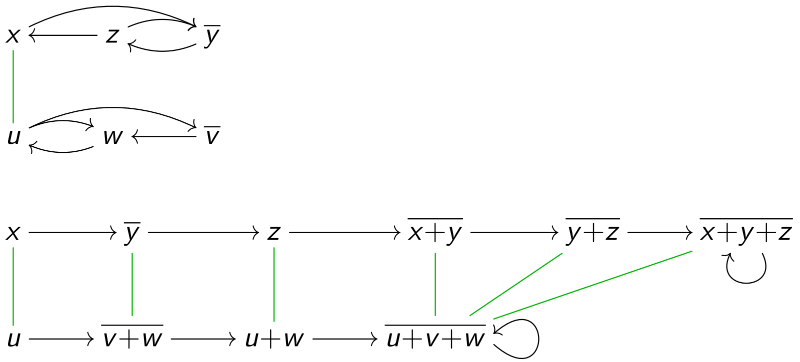

Non-Deterministic Automata

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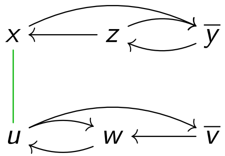

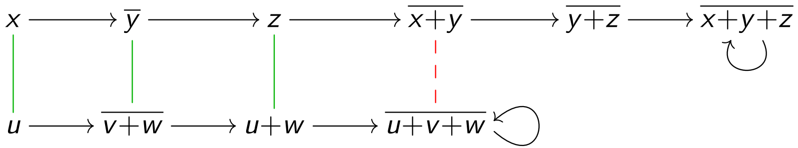

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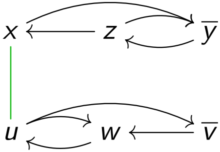
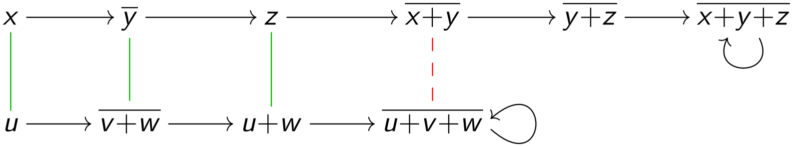


$$
\begin{array}{rl}
 & (x,\ u) \\
+ & (y,\ v{+}w) \\
\hline
= & (x{+}y,\ u{+}v{+}w)
\end{array}
$$



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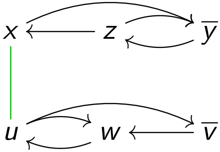
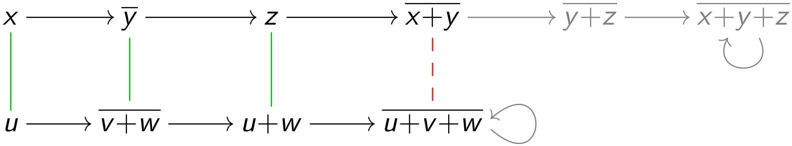


$$\begin{array}{rl} & (x,\ u) \\ + & (y,\ v{+}w) \\ \hline = & (x{+}y,\ u{+}v{+}w) \end{array}$$



using bisimulations up to union

Non-Deterministic Automata

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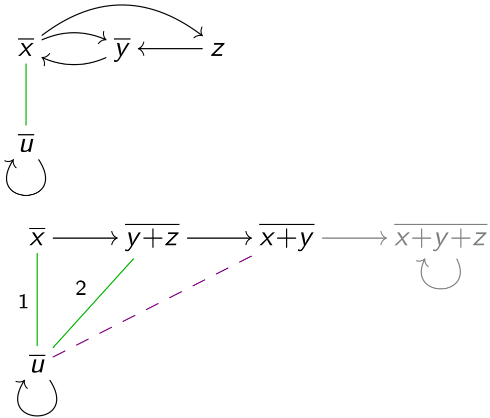

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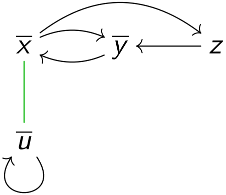
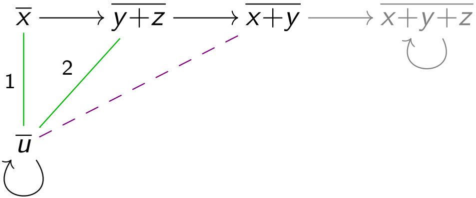


$$
\begin{aligned}
x+y &= u+y && (1)\\
&= y+z+y && (2)\\
&= y+z &&\\
&= u && (2)
\end{aligned}
$$



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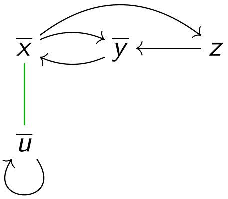

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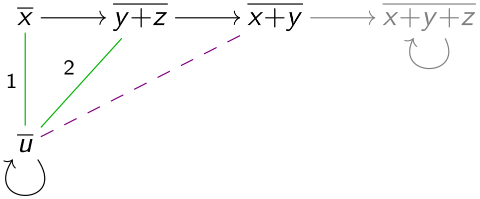

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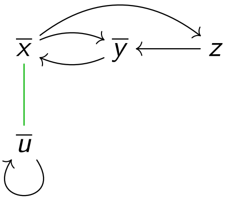

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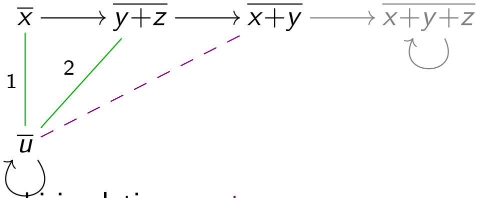

using bisimulations up to congruence

Non-Deterministic Automata

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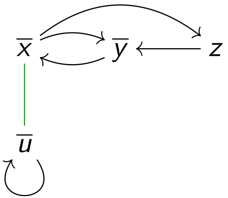

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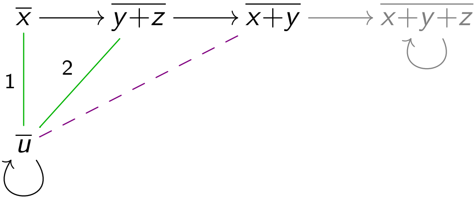

this yield to the HKC algorithm [Bonchi, Pous'13]

Outline

Up-to techniques at work

Deterministic finite automata

Non-Deterministic Automata

Stream calculus

Abstract coinduction in complete lattices

Coinductive stream calculus [Rutten'03]

Streams can be defined by behavioural differential equations:

$$
\begin{array}{lll}
(\sigma + \tau)' = \sigma' + \tau' & o(\sigma + \tau) = o(\sigma) + o(\tau) & \text{(sum)} \\
(\sigma \otimes \tau)' = \sigma' \otimes \tau + \sigma \otimes \tau' & o(\sigma \otimes \tau) = o(\sigma) \times o(\tau) & \text{(shuffle)} \\
(\sigma^{-1})' = -\sigma' \otimes (\sigma^{-1} \otimes \sigma^{-1}) & o(\sigma^{-1}) = o(\sigma)^{-1} & \text{(inverse)} \\
(i)' = 0 & o(i) = i & \text{(numbers)}
\end{array}
$$

A bisimulation is a relation R such that $\sigma \; R \; \tau$ entails $o(\sigma) = o(\tau)$ and $\sigma' \; R \; \tau'$

- Let us show that $\sigma + 0 \sim \sigma$

Coinductive stream calculus [Rutten'03]

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- How about $\sigma \otimes 1 \sim \sigma$?

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A bisimulation up to $\sim$ and is a relation R such that $\sigma\ R\ \tau$ entails $o(\sigma) = o(\tau)$ and $\sigma' \sim R \sim \tau'$

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- And $\sigma \otimes \sigma^{-1} \sim 1$?

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A bisimulation up to $\sim$ and up to context is a relation R such that $\sigma \; R \; \tau$ entails $o(\sigma) = o(\tau)$ and $\sigma' \sim c(R) \sim \tau'$

- Let us show that $\sigma + 0 \sim \sigma$
- How about $\sigma \otimes 1 \sim \sigma$?
- And $\sigma \otimes \sigma^{-1} \sim 1$?

Lessons learned from the examples

- A wide range of up-to techniques
  - up to equivalence
  - up to bisimilarity
  - up to union
  - up to context
- For different kind of systems
  - {deterministic,non-deterministic,(weighted)} automata,
  - streams
  - process algebra [Milner'89, Sangiorgi'98]
- Sometimes they need to be combined together
  - union and equivalence $\rightsquigarrow$ congruence (NFA)
  - c and $R \mapsto \sim R \sim$ $\rightsquigarrow$ $R \mapsto \sim c(R) \sim$ (streams)

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- ... but is this composition always sound?

Two questions

- Can we study all these proof principles in one framework?
- Can we derive conditions for soundness?

Compatiblity

Use Pous's algebra of enhancements: abstract framework in terms of lattices and monotone functions.

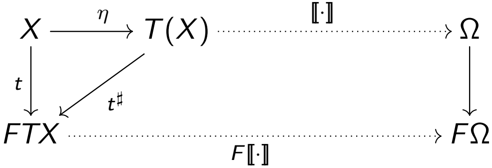
- b-simulation: $R \subseteq b(R)$;
- b-simulation up to f: $R \subseteq b(f(R))$
- Definition: f is b-compatible if $f \circ b \subseteq b \circ f$
- b-compatible functions: f sound and closed under composition

FT-Coalgebra

Coalgebras make it possible to encompass the previous examples in a uniform setting:

systems	functor (F)	monad (T)
deterministic automata	$2 \times -^A$	$(-)$
non-deterministic automata	$2 \times (-)^A$	$\mathcal{P}_{\mathsf{f}}(-)$
weigthed automata	$\mathbb{R} \times (-)^A$	$\mathbb{R}^{(-)}$
streams	$\mathbb{R} \times -$	$(-)$

First generalized powerset construction and then finality:



Behavioural equivalence becomes $x \sim_\alpha y \triangleq [\![\eta(x)]\!] = [\![\eta(y)]\!]$

Coalgebraic bisimulation

Given an F-coalgebra (X, α), define the following function on binary relations:

$$b_\alpha(R) = \{(x, y) \mid \exists z \in FR,\ F(\pi_1^R) = \alpha(x), F(\pi_2^R) = \alpha(y)\}$$

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Proposition [Rot, Bonchi, Bonsangue, Pous, Rutten, Silva'13]:
b_α satisfies (†) iff F preserves weak pullbacks

(†) $\forall R\, S, b(R) \cdot b(S) \subseteq b(R \cdot S)$

- up to equivalence (almost) always comes for free

Contexts: bialgebras

What about the up to union/context techniques?

- They are all instances of the same framework
  we just exploit some algebraic structure of the state-space:
    - a semilattice for non-deterministic automata
    - a vector space for weighted automata
    - a syntax for streams
- Can be captured using λ-bialgebras:

$$\lambda : TF \Rightarrow FT$$

$$TX \xrightarrow{\beta} X \xrightarrow{\alpha} FX$$

$$(\alpha \circ \beta = F\beta \circ \lambda_X \circ T\alpha)$$

[Turi&Plotkin'97, Bartels'04, Klin'11]

Summary

Coalgebras make it possible

- to exploit the abstract theory of up-to techniques for a wide range of systems
- to design algorithms in a uniform way

  (e.g., HKC for must-testing [Bonchi, Caltais, Pous, Silva'13])