Behavioural differential equations and coinduction for binary trees

Alexandra Silva and Jan Rutten

CWI, The Netherlands

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Motivation

- Previous work on streams and formal power series

  Behavioural differential equations: a coinductive calculus of streams, automata, and power series

  Elements of stream calculus (an extensive exercise in coinduction)

  showed that coinduction and behavioural differential equations are useful for building a calculus
- We want to investigate if the same approach is useful for other infinite structures, *e.g.* infinite binary trees

What will we show?

We will show how to…

- … define infinite binary trees through *behavioural differential equations*
- … develop a calculus for binary trees *à la formal power series*
- … calculate closed expressions for infinite binary trees
- … use closed expressions to prove properties about infinite binary trees

Infinite binary trees coalgebraically

Infinite binary trees

$$T_A = \{\sigma : \{L, R\}^* \to A\}$$

Recall: A formal power series is a function $\sigma : X^* \to A$ where X is the set of variables (or input symbols) and A is a semiring.

For A semiring, the set T_A coincides with the formal power series over $X = 2 = \{L, R\}$.

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Infinite binary trees coalgebraically

$(A^{B^*}, < o, t >)$ is the final coalgebra for functor $G(X) = A \times X^B$.

$$< o, t > \quad : \quad A^{B^*} \to A \times (A^{B^*})^B$$
$$< o, t > (f) \quad = \quad < f(\varepsilon), \lambda b \cdot \lambda w \cdot f(bw) >$$

$(T_A = A^{2^*}, \alpha)$ is the final coalgebra of $F(X) = A \times X^2$

$$\alpha(\sigma) = < \sigma(\varepsilon), \sigma_L, \sigma_R >$$

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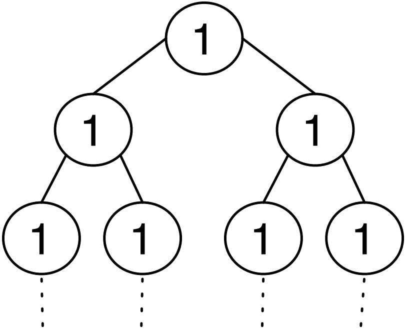
$\sigma \in T_A$ will be defined by means of **behavioural differential equations**:

$$\begin{array}{rcll} \sigma(\varepsilon) & = & c & \textit{initial value} \\ \sigma_L & = & \textit{left\_exp} & \textit{left derivative} \\ \sigma_R & = & \textit{right\_exp} & \textit{right derivative} \end{array}$$

Example I



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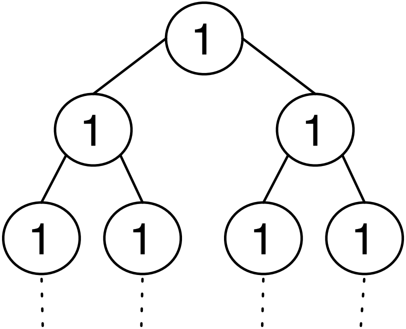

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Behavioural Differential Equations – format

In which conditions a system of behavioural differential equations defining a tree $\sigma = f(\sigma_1, \ldots, \sigma_n)$

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is well formed?

We proved that if:

1. c is calculated only involving $\sigma_1(\varepsilon), \ldots, \sigma_n(\varepsilon)$
2. left_exp and right_exp only depend on $\sigma_1, \ldots, \sigma_n, (\sigma_1)_L, \ldots, (\sigma_n)_L, (\sigma_1)_R, \ldots, (\sigma_n)_R$ and function symbols

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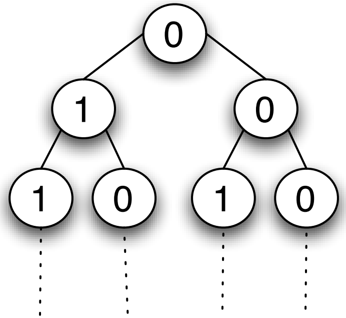

$$\sigma(\varepsilon) = 0$$
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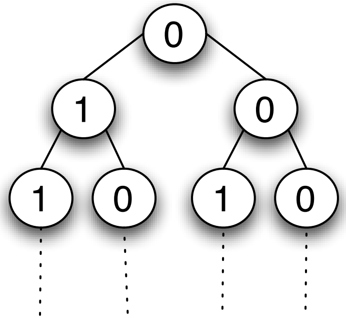

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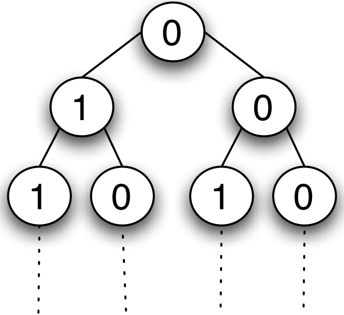

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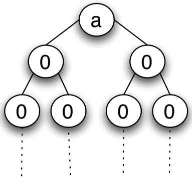
$[a]$ denotes



Examples III – The Thue-Morse sequence

- Obtained from the parities of the counts of 1's in the binary representation of non negative integers.
- 0,1,1,0,1,0,0,1, . . .
- Can be obtained by the substitution map $\{0 \to 01; 1 \to 10\}$:

$$0 \to 01 \to 0110 \to 01101001 \to \dots$$

- Tree representation (at level k, we have 2^k digits)

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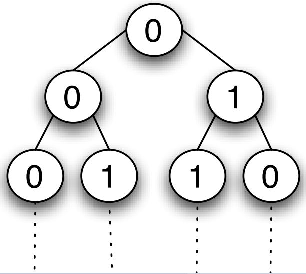
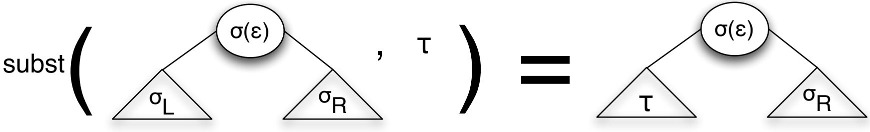


$$\begin{array}{rcl} thue(\varepsilon) & = & 0 \\ thue_L & = & thue \\ thue_R & = & thue + ones \end{array}$$

Examples IV – Substitution operation



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(subst(\sigma,\tau))(\varepsilon) &= \sigma(\varepsilon)\\
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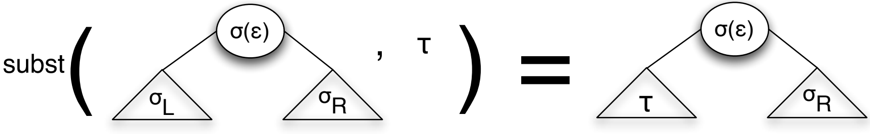

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Operations on trees

From formal power series we inherit several definitions of operations:

Name	Sum	Product
Initial value	$(\sigma+\tau)(\varepsilon)=\sigma(\varepsilon)+\tau(\varepsilon)$	$(\sigma \times \tau)(\varepsilon) = \sigma(\varepsilon) \times \tau(\varepsilon)$
Left der.	$(\sigma + \tau)_L = \sigma_L + \tau_L$	$(\sigma \times \tau)_L = \sigma_L \times \tau + \sigma(\varepsilon) \times \tau_L$
Right der	$(\sigma + \tau)_R = \sigma_R + \tau_R$	$(\sigma \times \tau)_R = \sigma_R \times \tau + \sigma(\varepsilon) \times \tau_R$

Fundamental Theorem

For all infinite binary trees $\sigma \in T_A$:

$$\sigma = \sigma(\varepsilon) + L \times \sigma_L + R \times \sigma_R$$

where

$$\begin{array}{ll} L(\varepsilon) = 0 & R(\varepsilon) = 0 \\ L_L = [1] & R_L = [0] \\ L_R = [0] & R_R = [1] \end{array}$$

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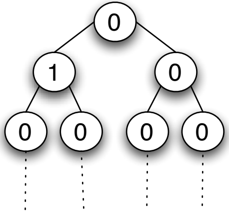
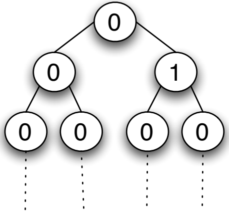
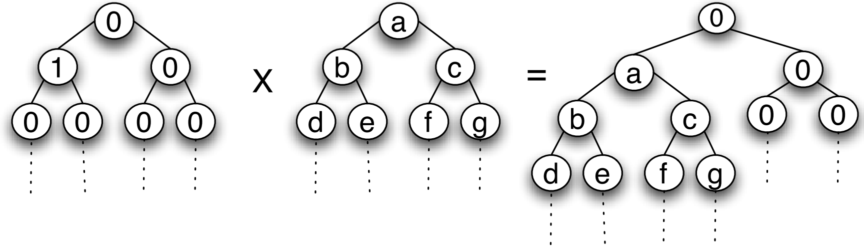


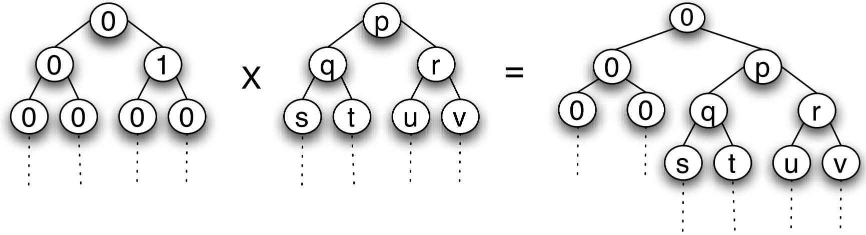


$L \times \sigma_L$



$R \times \sigma_R$

Similarly:



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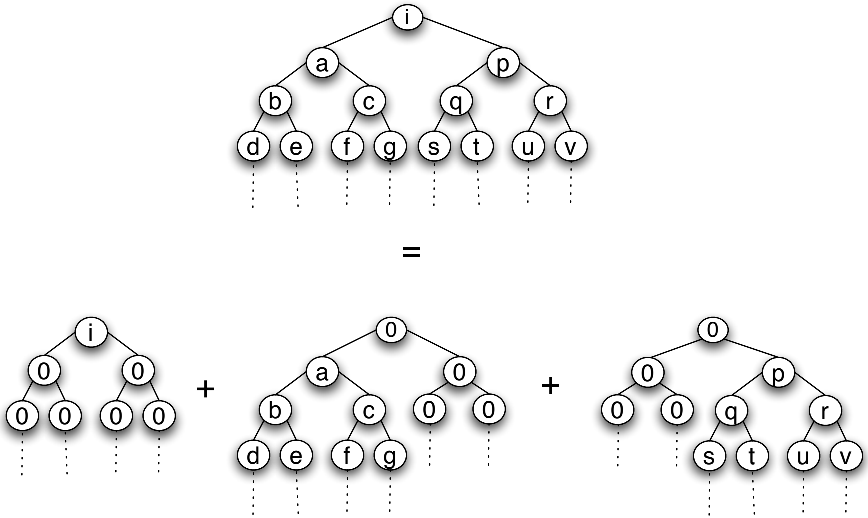
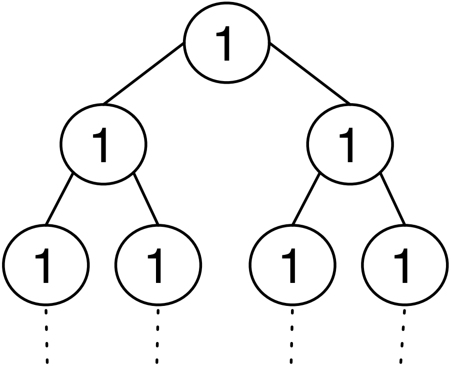


But. . . What can we do with this theorem?

?

Examples Revisited I



$$\begin{aligned} ones(\varepsilon) &= 1 \\ ones_L &= ones \\ ones_R &= ones \end{aligned}$$

Closed Formula I

$$\begin{aligned} & ones = 1 + L \times ones + R \times ones \\ \Leftrightarrow \quad & (1 - L - R) \times ones = 1 \\ \Leftrightarrow \quad & ones = \frac{1}{1-L-R} \end{aligned}$$

$\dfrac{x}{y}$ stands for $y^{-1} \times x$

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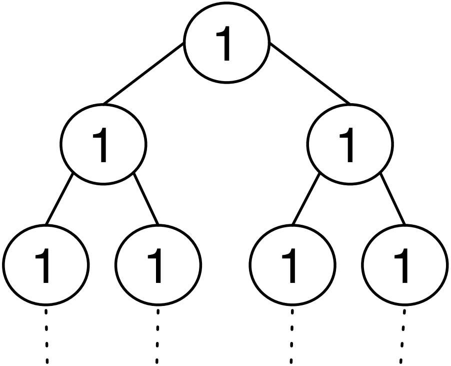

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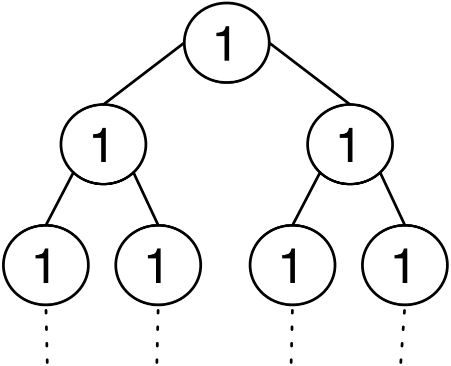

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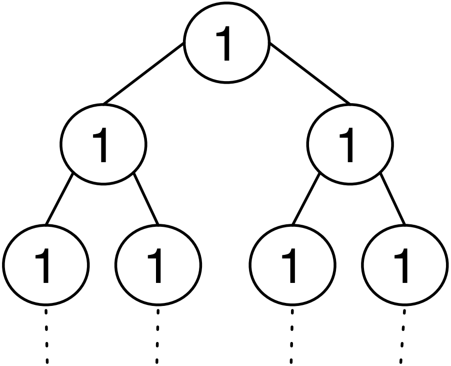

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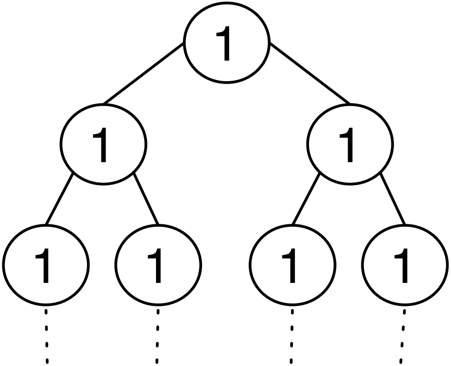

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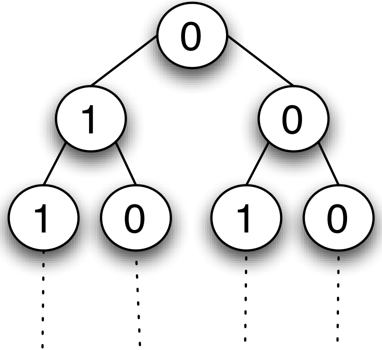
Inverse operation

The inverse of a tree – σ^{-1} – is defined formally so that $\sigma \times \sigma^{-1} = 1$.

$$\begin{array}{rcl} \sigma^{-1}(\varepsilon) & = & (\sigma(\varepsilon))^{-1} \\ (\sigma^{-1})_L & = & (\sigma(\varepsilon))^{-1} \times \sigma_L \times \sigma^{-1} \\ (\sigma^{-1})_R & = & (\sigma(\varepsilon))^{-1} \times \sigma_R \times \sigma^{-1} \end{array}$$

Examples Revisited II



$$\begin{aligned}\sigma(\varepsilon) &= 0\\ \sigma_L &= \sigma + 1\\ \sigma_R &= \sigma\end{aligned}$$

Closed Formula II

$$\begin{aligned} & \sigma = 0 + L \times (\sigma + 1) + R \times \sigma \\ \Leftrightarrow \quad & (1 - L - R)\sigma = L \\ \Leftrightarrow \quad & \sigma = \frac{L}{1-L-R}\end{aligned}$$

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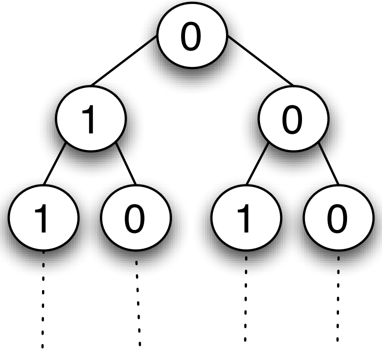

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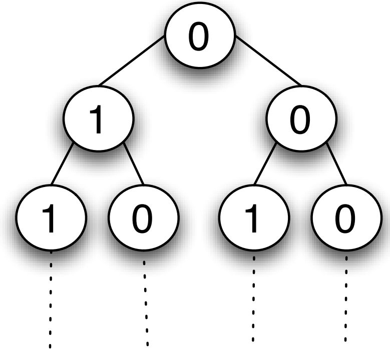

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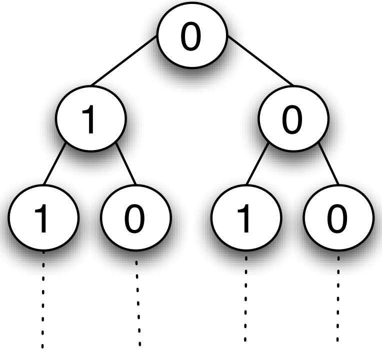

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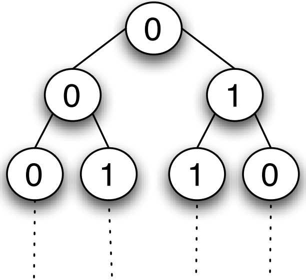

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Closed Formula III

$$\sigma = 0 + L \times \sigma + R \times (\sigma + \mathit{ones})$$

$$\Leftrightarrow \quad (1 - L - R)\sigma = R \times \frac{1}{1-L-R}$$

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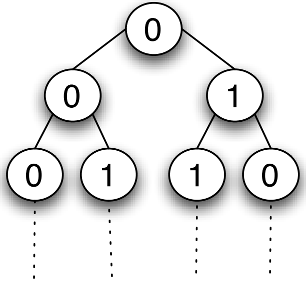

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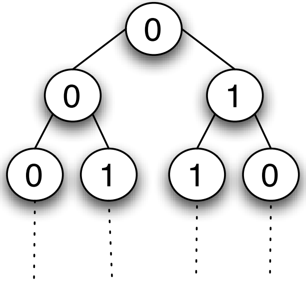

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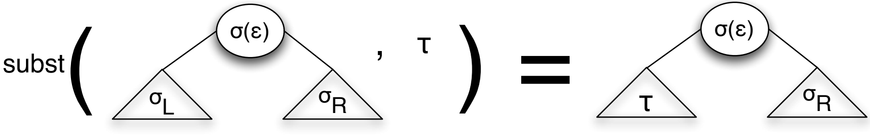

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Closed Formula IV

$$\mathit{subst}(\sigma,\tau) = \sigma(\varepsilon) + L \times \tau + R \times \sigma_R$$

$$\Leftrightarrow \quad \{\, \sigma - L \times \sigma_L = \sigma(\varepsilon) + R \times \sigma_R \}$$

$$\mathit{subst}(\sigma,\tau) = \sigma - L \times (\sigma_L - \tau)$$

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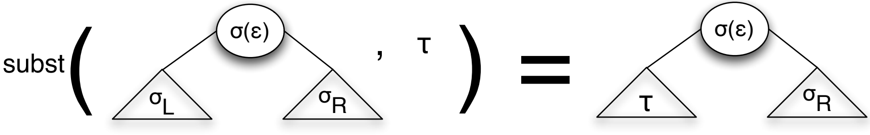

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(subst(\sigma,\tau))_R & = & \sigma_R
\end{array}
$$

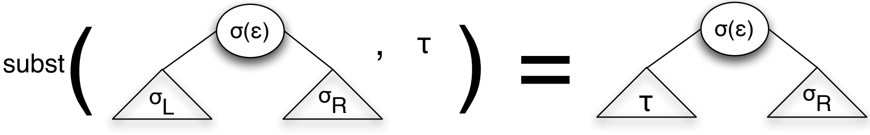
Closed Formula IV

$$subst(\sigma,\tau) = \sigma(\varepsilon) + L \times \tau + R \times \sigma_R$$

$$\Leftrightarrow \quad \{\sigma - L \times \sigma_L = \sigma(\varepsilon) + R \times \sigma_R\}$$

$$subst(\sigma,\tau) = \sigma - L \times (\sigma_L - \tau)$$

Examples revisited IV – Substitution



$$
\begin{aligned}
(\mathit{subst}(\sigma,\tau))(\varepsilon) &= \sigma(\varepsilon)\\
(\mathit{subst}(\sigma,\tau))_L &= \tau\\
(\mathit{subst}(\sigma,\tau))_R &= \sigma_R
\end{aligned}
$$

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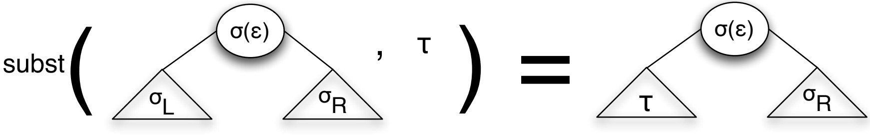

$$
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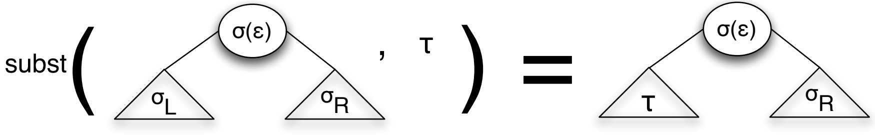

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\end{aligned}
$$

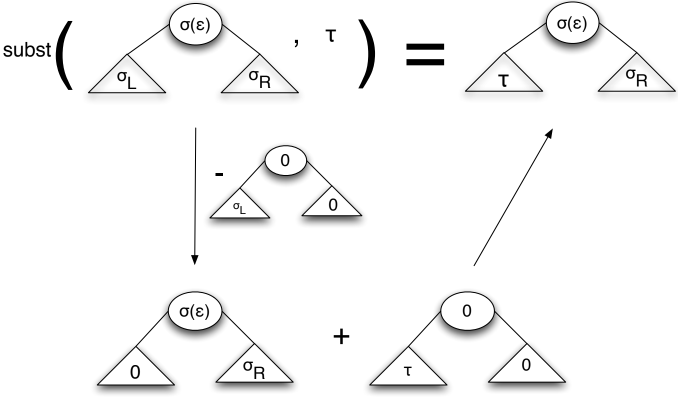
Closed Formula IV

$$
\mathit{subst}(\sigma,\tau) = \sigma(\varepsilon) + L \times \tau + R \times \sigma_R
$$

$$
\Leftrightarrow \qquad \{\, \sigma - L \times \sigma_L = \sigma(\varepsilon) + R \times \sigma_R \}
$$

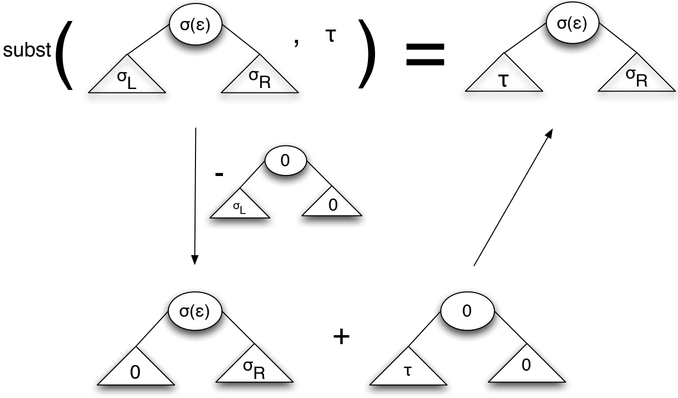
$$
\mathit{subst}(\sigma,\tau) = \sigma - L \times (\sigma_L - \tau)
$$

$subst(\sigma, \tau) = \sigma - L \times (\sigma_L - \tau)$



Easily generalizes: $subst(\sigma, \tau, P) = \sigma - P(\sigma_P - \tau)$

$subst(\sigma, \tau) = \sigma - L \times (\sigma_L - \tau)$



Easily generalizes: $subst(\sigma, \tau, P) = \sigma - P(\sigma_P - \tau)$

Proving properties

subst$(\sigma, \sigma_P, P) = \sigma$ and *subst*$(subst(\sigma, \tau, P), \sigma_P, P) = \sigma$

- **Property 1**

$$subst(\sigma, \sigma_P, P) = \sigma - P(\sigma_P - \sigma_P) = \sigma$$

- **Property 2**

$$\begin{aligned} & subst(subst(\sigma, \tau, P), \sigma_P, P) & \\ = \; & subst(\sigma - P(\sigma_P - \tau), \sigma_P, P) & \text{Def. of } subst \\ = \; & \sigma - P(\sigma_P - \tau) - P((\sigma - P(\sigma_P - \tau))_P - \sigma_P) & \text{Def. of } subst \\ = \; & \sigma - P(\sigma_P - \tau) - P(\tau - \sigma_P) & \\ = \; & \sigma & \end{aligned}$$

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Conclusions

- Modelling binary trees as formal power series we have derived a calculus
- We have defined various non-trivial trees by means of simple differential equations
- We have showed how to compute compact closed formulae for them
- We have also illustrated that this can be used to prove properties about trees

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Future work

- Behavioural differential equations are closely related to lazy functional programming implementations.
- In particular, we would like to study the relation between closed expressions and elimination of corecursion
- The closed formula that we have obtained for the Thue-Morse sequence suggests a possible use of coinduction and differential equations in the area of automatic sequences.