Algebra-Coalgebra Duality: applications in automata theory

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The global message

- Two views on many problems: Algebra and coalgebra.
- The combination is essential!
- Coalgebra is semantics but also algorithms.

(Co)algebra

Specify and reason about systems.

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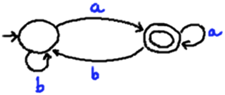
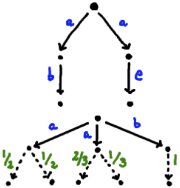
state-machines
e.g. DFA, LTS, PA





(Co)algebra

Specify and **reason** about **systems**.

Syntax
RE, CCS, …

$b^*a(b^*a)^*$

$a.b.0 + a.c.0$

$a.(\frac{1}{2}\cdot 0 \oplus \frac{1}{2}\cdot 0) + \cdots$

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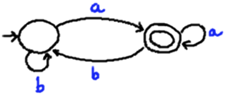

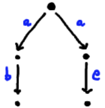

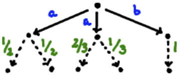

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Axiomatization
KA,...

$1 + a a^* = a^*$
$\vdots$

$P + 0 = P$
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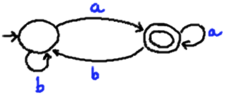

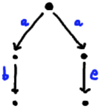

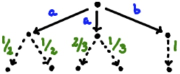

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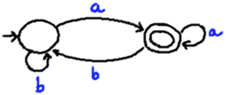
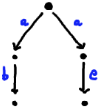
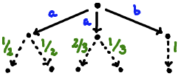
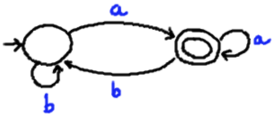


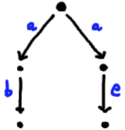


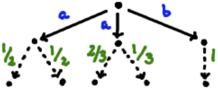


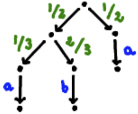
Can we do all of this uniformly in a single framework?

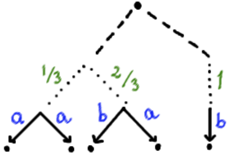
What do this things have in common?











$(S, t : S \to 2 \times S^A)$

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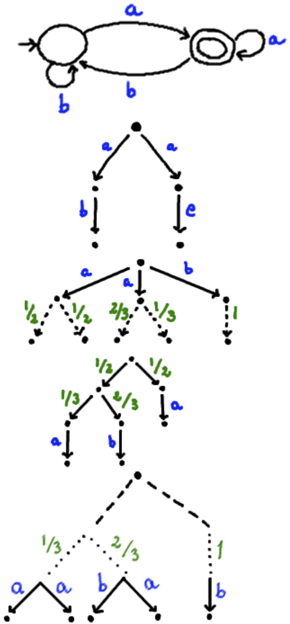

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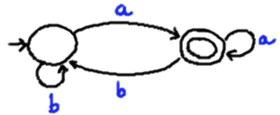

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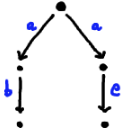

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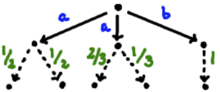

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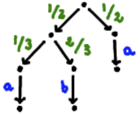

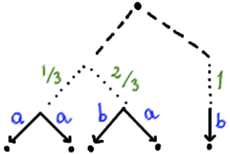

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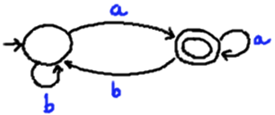

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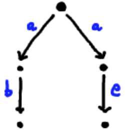

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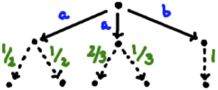

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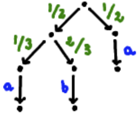

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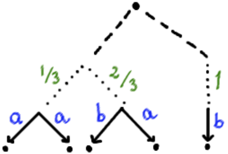

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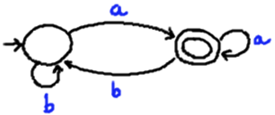

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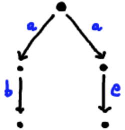

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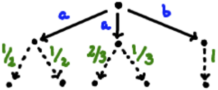

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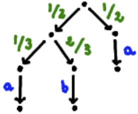

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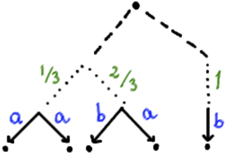

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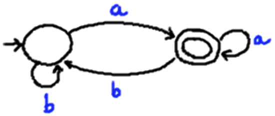

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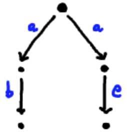

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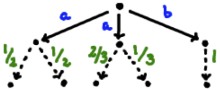

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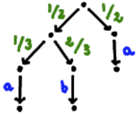

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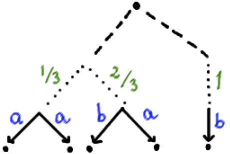

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$$(S, t : S \to TS)$$

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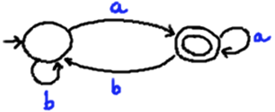

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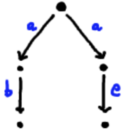

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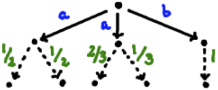

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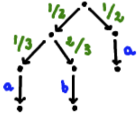

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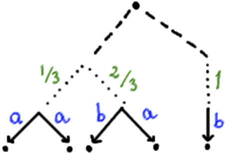

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$(S, t : S \to TS)$ T-coalgebras

The power of T

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3. set of expressions describing finite systems
4. axioms to prove bisimulation equivalence of expressions

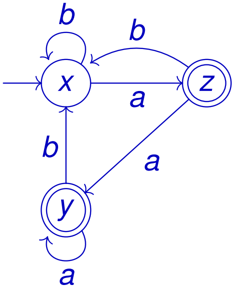
1 + 2 are classic coalgebra; 3 + 4 are recent work.

Brzozowski's algorithm (co)algebraically

Motivation

- duality between reachability and observability (Arbib and Manes 1975): beautiful, not very well-known.
- combined use of algebra and coalgebra.
- our understanding of automata is still very limited;
  cf. recent research: universal automata, àtomata, weighted automata (Brzozowski, Ésik, Sakarovitch, . . .)

Brzozowski algorithm (by example)



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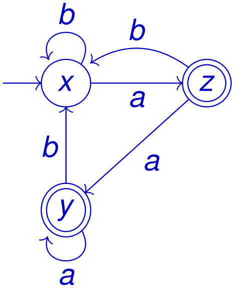
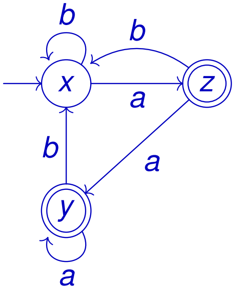


- initial state: x • final states: y and z
- $L(x) = \{a, b\}^* a$
- X is reachable but not minimal: $L(y) = \varepsilon + \{a, b\}^* a = L(z)$

Reversing the automaton: $rev(X)$

$X =$ 

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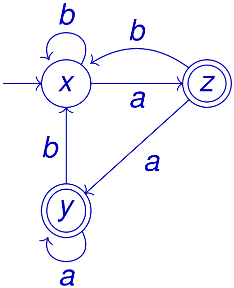

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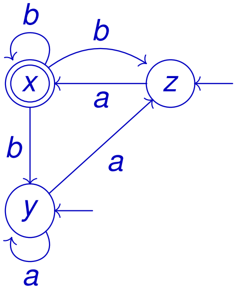

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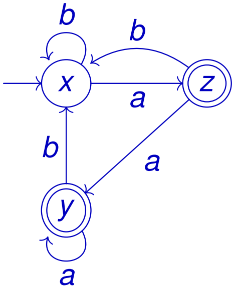

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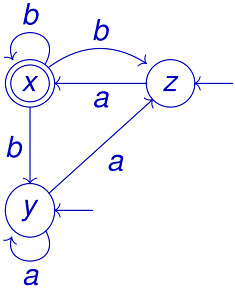

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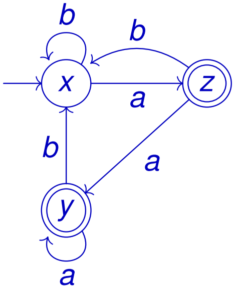

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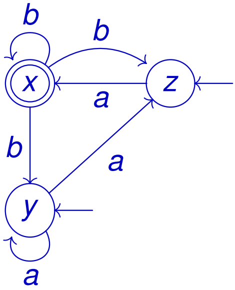
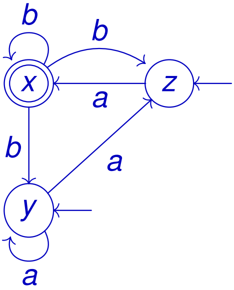


- transitions are reversed
- initial states $\Leftrightarrow$ final states
- $rev(X)$ is non-deterministic

Making it deterministic again: $det(rev(X))$



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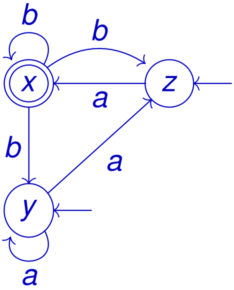

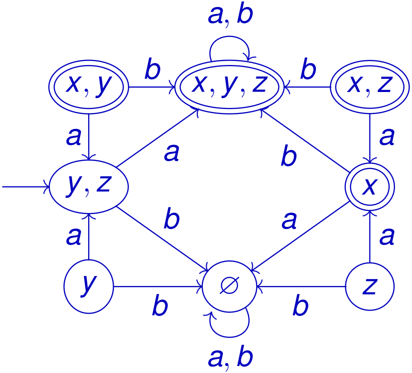

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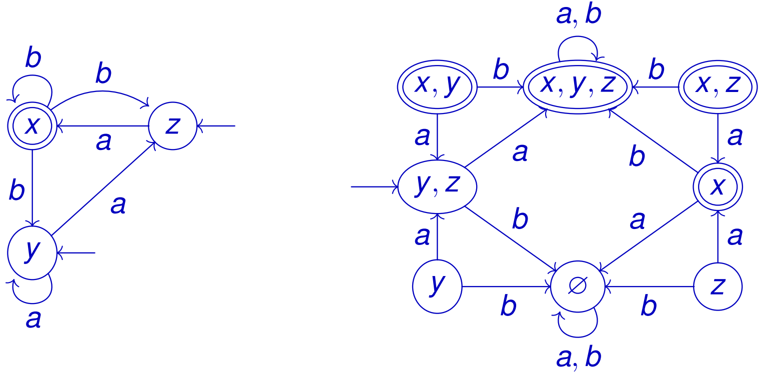

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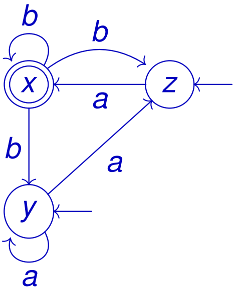
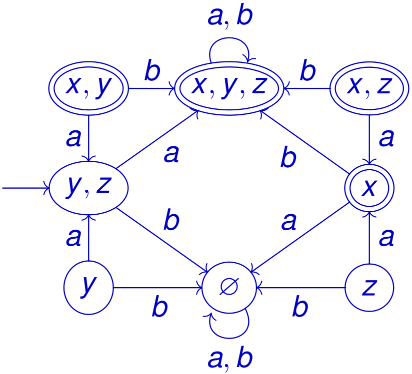
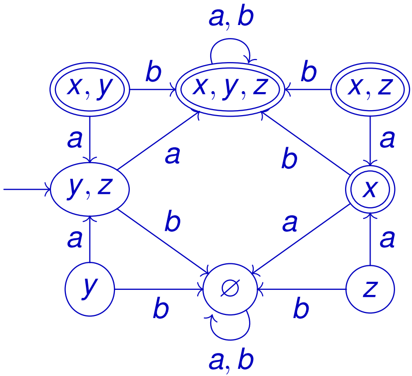




- new state space: $2^X = \{V \mid V \subseteq \{x, y, z\}\}$
- initial state: $\{y, z\}$ final states: all V with $x \in V$
- $V \xrightarrow{a} W \qquad W = \{w \mid v \xrightarrow{a} w, v \in V\}$

The automaton $det(rev(X))$ . . .



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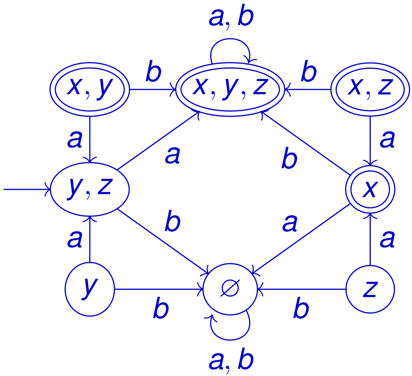

- . . . accepts the reverse of the language accepted by X:

$$L(\,det(rev(X))\,) \;=\; a\,\{a, b\}^* \;=\; reverse(\,L(X)\,)$$

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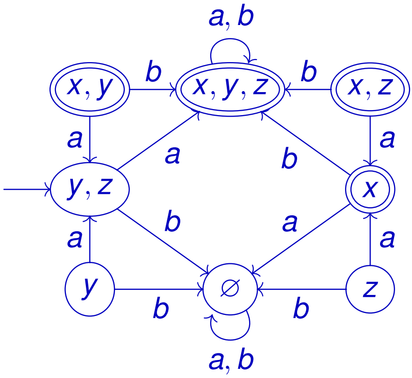


- . . . accepts the reverse of the language accepted by X:

$$L(\,det(rev(X))\,) \;=\; a\,\{a, b\}^* \;=\; reverse(\,L(X)\,)$$

- . . . and is observable!

Today's Theorem

If: a deterministic automaton X is reachable and accepts $L(X)$

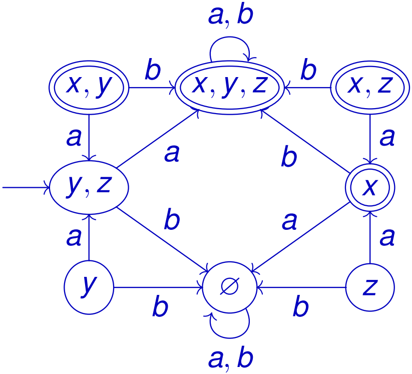
Today's Theorem

If: a deterministic automaton X is reachable and accepts $L(X)$

then: $det(rev(X))$ is minimal and

$$L(\,det(rev(X))\,) \;=\; reverse(\,L(X)\,)$$

Taking the reachable part of $det(rev(X))$



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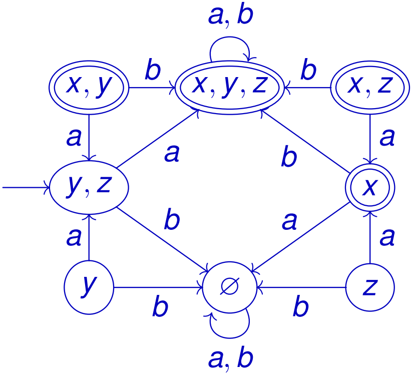
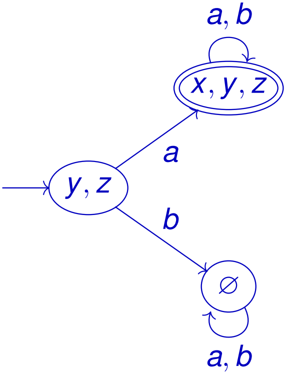
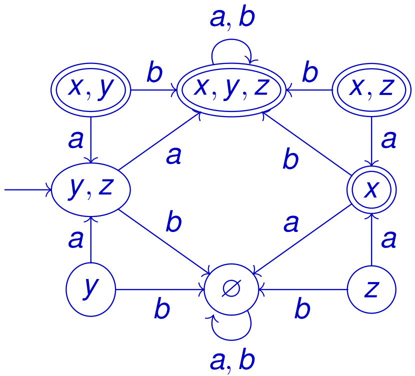
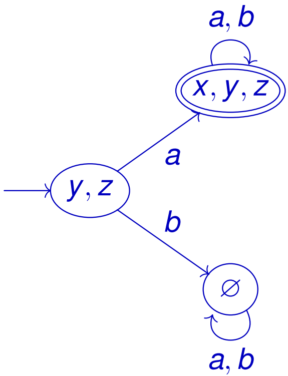




- $reach(det(rev(X)))$

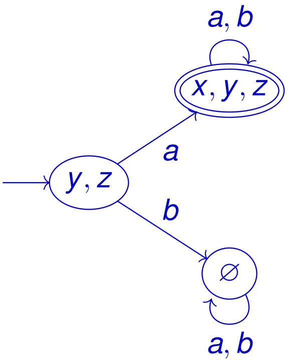
Taking the reachable part of *det*(*rev*(*X*))





- $reach(det(rev(X)))$ is reachable (by construction)

Repeating everything, now for *reach*(*det*(*rev*(X)))



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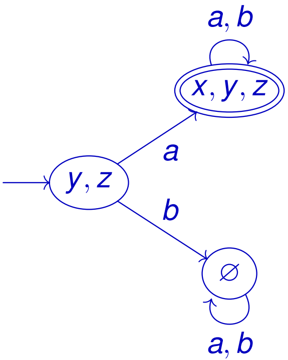
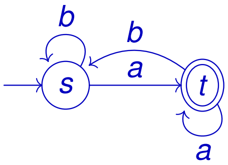
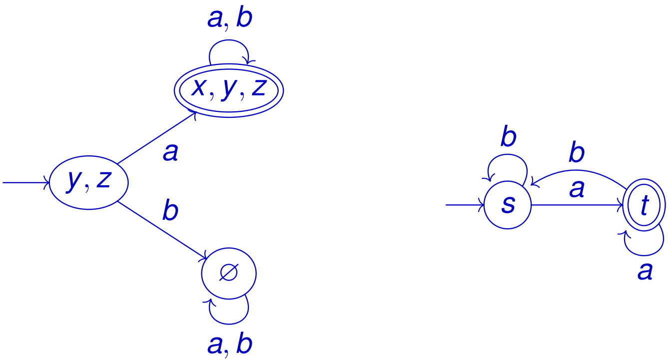


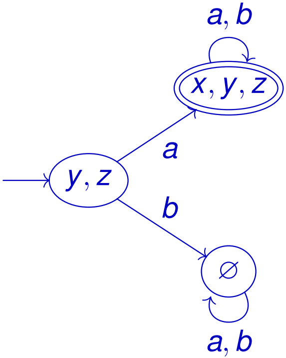


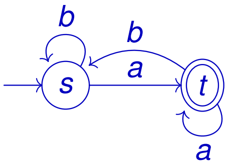
Repeating everything, now for *reach*(*det*(*rev*(*X*)))



- . . . gives us *reach*(*det*(*rev*(*reach*(*det*(*rev*(*X*))))))

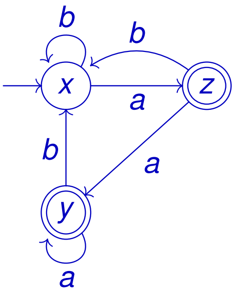
Repeating everything, now for *reach*(*det*(*rev*(*X*)))





- . . . gives us $reach(det(rev(reach(det(rev(X))))))$
- which is (reachable and) minimal and accepts $\{a, b\}^*\, a$.

All in all: Brzozowski's algorithm



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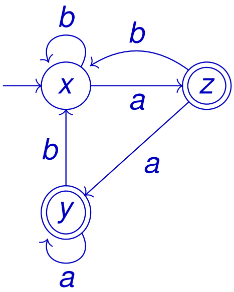

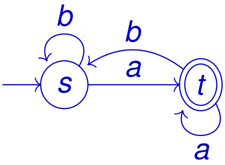

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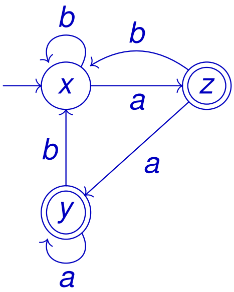
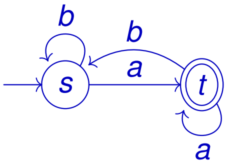
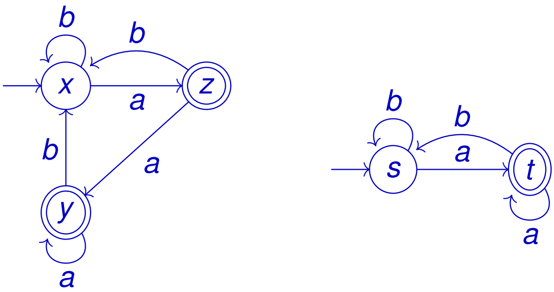




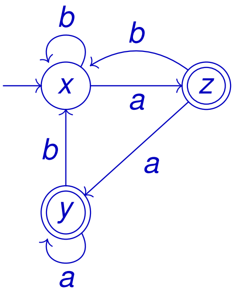
- X is reachable and accepts $\{a, b\}^* \, a$

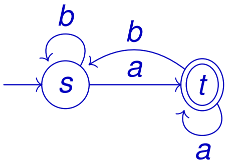
All in all: Brzozowski's algorithm



- X is reachable and accepts $\{a, b\}^* a$
- $reach(det(rev(reach(det(rev(X))))))$ also accepts $\{a, b\}^* a$

All in all: Brzozowski's algorithm





- X is reachable and accepts $\{a, b\}^*\ a$
- $reach(det(rev(reach(det(rev(X))))))$ also accepts $\{a, b\}^*\ a$
- ... and is minimal!!

Goal of the day

- Correctness of Brzozowski's algorithm (co)algebraically
- Generalizations to other types of automata

(Co)algebra

algebras:
$$\begin{array}{c} F(X) \\ \downarrow f \\ X \end{array}$$

coalgebras:
$$\begin{array}{c} X \\ \downarrow f \\ F(X) \end{array}$$

Examples of algebras

$$\begin{tikzcd} \mathbb{N} \times \mathbb{N} \arrow[d, "+"'] \\ \mathbb{N} \end{tikzcd}$$

Examples of algebras

$$
\begin{tikzcd}
\mathbb{N} \times \mathbb{N} \arrow[d, "+"'] \\
\mathbb{N}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
1 + \mathbb{N} \arrow[d, "{[0, S]}"'] \\
\mathbb{N}
\end{tikzcd}
\equiv
\begin{tikzcd}
1 \arrow[dr, "0"'] & & \mathbb{N} \arrow[dl, "S"] \\
& \mathbb{N} &
\end{tikzcd}
\equiv
\begin{tikzcd}
1 \arrow[dr, "0"'] & \\
& \mathbb{N} \arrow[d, "S"] \\
& \mathbb{N}
\end{tikzcd}
$$

Examples of coalgebras

$$
\begin{tikzcd}
X \arrow[d, "t"'] \\
\mathcal{P}(A \times X)
\end{tikzcd}
\qquad\qquad
x \xrightarrow{a} y \quad \leftrightarrow \; \langle a, y \rangle \in t(x)
$$

Examples of coalgebras

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\end{tikzcd}
\qquad
x \xrightarrow{a} y \quad \leftrightarrow \quad \langle a, y \rangle \in t(x)
$$

$$
\begin{tikzcd}
X \arrow[d, "{\langle Left, label, Right \rangle}"'] \\
X \times A \times X
\end{tikzcd}
$$

Examples of coalgebras

$$
\begin{tikzcd}
2^\omega \arrow[d, "\langle \text{head}, \text{tail} \rangle"'] \\
2 \times 2^\omega
\end{tikzcd}
\quad \equiv \quad
\begin{tikzcd}
& 2^\omega \arrow[dl, "\text{head}"'] \arrow[dr, "\text{tail}"] & \\
2 & & 2^\omega
\end{tikzcd}
\quad \equiv \quad
\begin{tikzcd}
& 2 \\
2^\omega \arrow[ur, "\text{head}"] \arrow[d, "\text{tail}"] & \\
2^\omega &
\end{tikzcd}
$$

$$\text{head}((b_0, b_1, b_2, \ldots)) = b_0$$

$$\text{tail}((b_0, b_1, b_2, \ldots)) = (b_1, b_2, b_3 \ldots)$$

Homomorphisms

$$\begin{tikzcd} F(X) \arrow[r, "F(h)"] \arrow[d, "f"'] & F(Y) \arrow[d, "g"] \\ X \arrow[r, "h"'] & Y \end{tikzcd}$$

Homomorphisms

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Initiality, finality

$$\begin{tikzcd} F(A) \arrow[r, dashed, "F(h)"] \arrow[d, "\alpha"'] & F(X) \arrow[d, "f"] \\ A \arrow[r, dashed, "\exists!\, h"'] & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, dashed, "\exists!\, h"] \arrow[d, "f"'] & Z \arrow[d, "\beta"] \\ F(X) \arrow[r, dashed, "F(h)"'] & F(Z) \end{tikzcd}$$

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- initial algebras $\leftrightarrow$ induction

Initiality, finality

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- initial algebras $\leftrightarrow$ induction
- final coalgebras $\leftrightarrow$ coinduction

Automata, (co)algebraically

- Automata are complicated structures:

  part of them is algebra - part of them is coalgebra

Automata, (co)algebraically

- Automata are complicated structures:

  part of them is algebra - part of them is coalgebra
- (. . . in two different ways . . .)

A deterministic automaton

$$\begin{tikzcd}
1 \arrow[dr, "i"] & & 2 \\
& X \arrow[ur, "f"] \arrow[d, "t"] & \\
& X^A &
\end{tikzcd}$$

A deterministic automaton

$$\begin{tikzcd}
1 \arrow[dr, "i"] & & 2 \\
& X \arrow[ur, "f"'] \arrow[d, "t"] & \\
& X^A &
\end{tikzcd}$$

where

$$1 = \{0\} \qquad 2 = \{0, 1\} \qquad X^A = \{\, g \mid g : A \to X \,\}$$

$$x \xrightarrow{a} y \quad \leftrightarrow \quad t(x)(a) = y$$

$i(0) \in X$ is the initial state

x is final (or accepting) $\leftrightarrow$ $f(x) = 1$

Automata: algebra or coalgebra?

- initial state: algebraic – final states: coalgebraic

$$\begin{tikzcd} 1 \arrow[dr, "i"] & & 2 \\ & X \arrow[ur, "f"] & \end{tikzcd}$$

Automata: algebra or coalgebra?

- initial state: algebraic – final states: coalgebraic

$$
\begin{tikzcd}
1 \arrow[dr, "i"] & & 2 \\
& X \arrow[ur, "f"] &
\end{tikzcd}
$$

- transition function: both algebraic and coalgebraic

$$
\frac{\dfrac{X \xrightarrow{t} X^A}{X \longrightarrow (A \longrightarrow X)}}{X \times A \xrightarrow{t} X}
$$

Automata: algebra and coalgebra!

$$
\begin{tikzcd}
1 \arrow[d, "\epsilon"'] \arrow[dr, "i"] & & 2 \\
A^* \arrow[r, dashed, "r"'] \arrow[d, "\alpha"'] & X \arrow[r, dashed, "o"'] \arrow[ur, "f"] \arrow[d, "t"] & 2^{A^*} \arrow[u, "\epsilon?"'] \arrow[d, "\beta"] \\
(A^*)^A \arrow[r, dashed, "r^A"'] & X^A \arrow[r, dashed, "o^A"'] & (2^{A^*})^A
\end{tikzcd}
$$

Automata: algebra and coalgebra!

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To take home: this picture!! . . .

Automata: algebra and coalgebra!

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\end{tikzcd}
$$

To take home: this picture!! . . . which we'll explain next . . .

The “automaton” of languages

$$
\begin{tikzcd}
2 \\
2^{A^*} \arrow[u, "\epsilon?"'] \arrow[d, "\beta"] \\
(2^{A^*})^A
\end{tikzcd}
$$

$$\epsilon?(L) = 1 \leftrightarrow \epsilon \in L$$

$$2^{A^*} = \{g \mid g : A^* \to 2\} \cong \{L \mid L \subseteq A^*\}$$

$$\beta(L)(a) = L_a = \{w \in A^* \mid a \cdot w \in L\}$$

The "automaton" of languages

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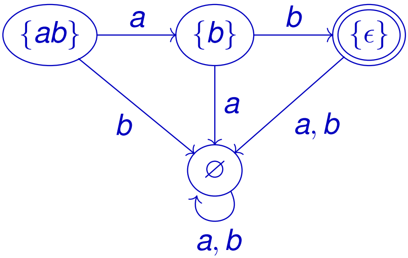
- We say "automaton": it does not have an initial state.

The automaton of languages

- transitions: $L \xrightarrow{a} L_a$ where $L_a = \{w \in A^* \mid a \cdot w \in L\}$
- for instance:



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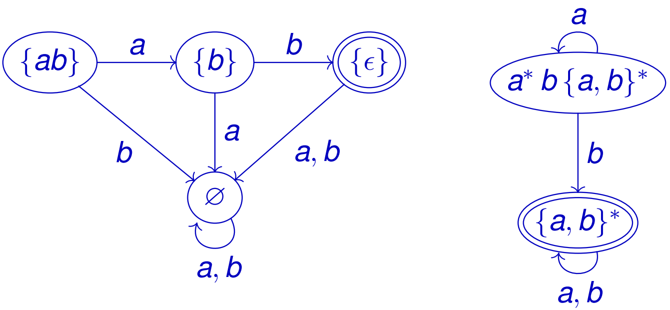

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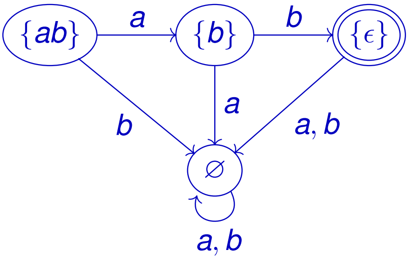

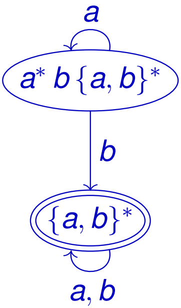

- note: every state L accepts . . .

The automaton of languages

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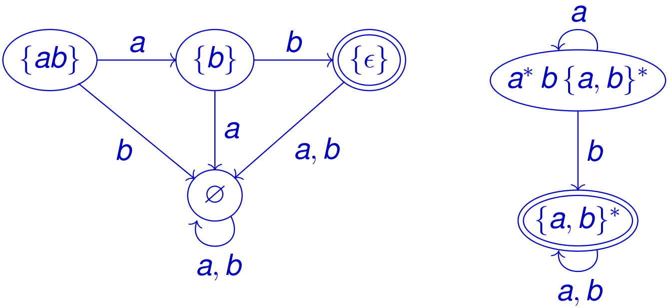

- note: every state L accepts . . . . . . the language L !!

The automaton of languages is . . . final

$$
\begin{tikzcd}
 & 2 \\
X \arrow[ur, "f"] \arrow[r, dashed, "\exists!\, o"'] \arrow[d, "t"] & 2^{A^*} \arrow[u, "\epsilon?"'] \arrow[d, "\beta"] \\
X^A \arrow[r, dashed, "o^A"'] & (2^{A^*})^A
\end{tikzcd}
$$

$$
\begin{aligned}
o(x) &= \{w \in A^* \mid f(x_w) = 1\} \\
&= \text{the language accepted by } x
\end{aligned}
$$

The automaton of languages is . . . final

$$\begin{tikzcd}
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\end{tikzcd}$$

$$\begin{aligned} o(x) &= \{ w \in A^* \mid f(x_w) = 1 \} \\ &= \text{the language accepted by } x \end{aligned}$$

where: x_w is the state reached after inputting the word w,

and: $o^A(g) = o \circ g$, all $g \in X^A$.

Back to today's picture

$$
\begin{tikzcd}
1 \arrow[d, "\epsilon"'] \arrow[dr, "i"] & & 2 \\
A^* \arrow[r, dashed, "r"'] \arrow[d, "\alpha"'] & X \arrow[r, dashed, "o"'] \arrow[ur, "f"] \arrow[d, "t"] & 2^{A^*} \arrow[u, "\epsilon?"'] \arrow[d, "\beta"] \\
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Back to today's picture

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\end{tikzcd}
$$

On the right: final coalgebra

Back to today's picture

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\begin{tikzcd}
1 \arrow[d, "\epsilon"'] \arrow[dr, "i"] & & 2 \\
A^* \arrow[r, dashed, "r"'] \arrow[d, "\alpha"'] & X \arrow[r, dashed, "o"'] \arrow[ur, "f"] \arrow[d, "t"] & 2^{A^*} \arrow[u, "\epsilon?"'] \arrow[d, "\beta"] \\
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\end{tikzcd}
$$

On the right: final coalgebra

On the left: initial algebra . . .

The "automaton" of words

$$\begin{tikzcd} 1 \arrow[d, "\epsilon"'] \\ A^* \arrow[d, "\alpha"'] \\ (A^*)^A \end{tikzcd}$$

ϵ is initial state

$$\alpha(w)(a) = w \cdot a$$

that is, transitions: $w \xrightarrow{a} w \cdot a$

The automaton of words is . . . initial

$$\begin{tikzcd}
1 \arrow[d, "\epsilon"'] \arrow[dr, "i"] & \\
A^* \arrow[r, dashed, "\exists!\, r"'] \arrow[d, "\alpha"'] & X \arrow[d, "t"] \\
(A^*)^A \arrow[r, dashed, "r^A"'] & X^A
\end{tikzcd}$$

$$\begin{aligned}
i \in X &= \text{initial state} \\
& \text{(to be precise: } i(0)\text{)} \\
r(w) &= i_w \\
&= \text{the state reached from } i \\
& \text{after inputting } w
\end{aligned}$$

- Proof: easy exercise.
- Proof: formally, because A^* is an initial $1 + A \times (-)$-algebra!

Duality

- Reachability and observability are dual:

  Arbib and Manes, 1975.
- (here observable = minimal)

Reachability and observability

$$\begin{tikzcd}
1 \arrow[d, "\epsilon"'] \arrow[dr, "i"] & & 2 \\
A^* \arrow[r, dashed, "r"'] \arrow[d, "\alpha"'] & X \arrow[r, dashed, "o"'] \arrow[ur, "f"] \arrow[d, "t"] & 2^{A^*} \arrow[u, "\epsilon?"'] \arrow[d, "\beta"] \\
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\end{tikzcd}$$

Reachability and observability

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\end{tikzcd}
$$

$r(w) \;=\;$ state reached on input w

$o(x) \;=\;$ language accepted by x

Reachability and observability

$$
\begin{tikzcd}
1 \arrow[d, "\epsilon"'] \arrow[dr, "i"] & & 2 \\
A^* \arrow[r, dashed, "r"'] \arrow[d, "\alpha"'] & X \arrow[r, dashed, "o"'] \arrow[ur, "f"] \arrow[d, "t"] & 2^{A^*} \arrow[u, "\epsilon?"'] \arrow[d, "\beta"] \\
(A^*)^A \arrow[r, dashed, "r^A"'] & X^A \arrow[r, dashed, "o^A"'] & (2^{A^*})^A
\end{tikzcd}
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\begin{aligned}
r(w) &= \text{state reached on input } w \\
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- We call X reachable if r is surjective.

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$$
\begin{aligned}
r(w) &= \text{state reached on input } w \\
o(x) &= \text{language accepted by } x
\end{aligned}
$$

- We call X reachable if r is surjective.
- We call X observable (= minimal) if o is injective.

Reversing the automaton

- Reachability $\leftrightarrow$ observability
- Being precise about homomorphisms is crucial.
- Forms the basis for proof Brzozowski's algorithm.

Powerset construction

$$2^{(-)} : \quad \begin{tikzcd} V \arrow[d, "g"'] \\ W \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} 2^V \\ 2^W \arrow[u, "2^g"'] \end{tikzcd}$$

Powerset construction

$$\begin{tikzcd} 2^{(-)} \, : & V \arrow[d, "g"'] & \mapsto & 2^V \\ & W & & 2^W \arrow[u, "2^g"'] \end{tikzcd}$$

where $2^V = \{S \mid S \subseteq V\}$ and, for all $S \subseteq W$,

$$2^g(S) = g^{-1}(S) \quad (= \{v \in V \mid g(v) \in S\})$$

Powerset construction

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- This construction is contravariant !!

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where $2^V = \{S \mid S \subseteq V\}$ and, for all $S \subseteq W$,

$$2^g(S) = g^{-1}(S) \quad (= \{v \in V \mid g(v) \in S\})$$

- This construction is contravariant !!
- Note: if g is surjective, then 2^g is injective.

Reversing transitions

$$\begin{array}{c} X \\ \Big\downarrow t \\ X^A \end{array}$$

Reversing transitions

$$
\begin{array}{c||c}
X & X \times A \\
{\scriptstyle t}\big\downarrow & \big\downarrow \\
X^A & X
\end{array}
$$

Reversing transitions

$$\begin{tikzcd}[row sep=large] X \arrow[d, "t"'] & \Big\| & X \times A \arrow[d] & & 2^{X \times A} \\ X^A & \Big\| & X \arrow[r, phantom, "\overset{2^{(-)}}{\longmapsto}"] & & 2^X \arrow[u] \end{tikzcd}$$

Reversing transitions

$$
\begin{array}{c|c}
X & X \times A \\
t \downarrow & \downarrow \\
X^A & X
\end{array}
\quad \xmapsto{2^{(-)}} \quad
\begin{array}{c|c}
2^{X \times A} & (2^X)^A \\
\uparrow & \uparrow \\
2^X & 2^X
\end{array}
$$

Reversing transitions

$$
\begin{tikzcd}
X \arrow[d, "t"'] \\
X^A
\end{tikzcd}
\quad \Bigg\| \quad
\begin{tikzcd}
X \times A \arrow[d] \\
X
\end{tikzcd}
\quad \xmapsto{\;2^{(-)}\;} \quad
\begin{tikzcd}
2^{X \times A} \\
2^X \arrow[u]
\end{tikzcd}
\quad \Bigg\| \quad
\begin{tikzcd}
(2^X)^A \\
2^X \arrow[u]
\end{tikzcd}
\quad \Bigg\| \quad
\begin{tikzcd}
2^X \arrow[d, "2^t"] \\
(2^X)^A
\end{tikzcd}
$$

Initial $\leftrightarrow$ final

$$\begin{tikzcd} 1 \arrow[dr, "i"] & \\ & X \end{tikzcd}$$

Initial $\leftrightarrow$ final

$$\begin{tikzcd} 1 \arrow[dr, "i"] & & & & 2 \\ & X & \arrow[r, mapsto, "2^{(-)}"] & 2^X \arrow[ur, "2^i"] & \end{tikzcd}$$

Initial $\leftrightarrow$ final

$$
\begin{tikzcd}
1 \arrow[dr, "i"] & & & & & & 2 \\
& X & & \arrow[r, mapsto, "2^{(-)}"] & {} & 2^X \arrow[ur, "2^i"] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& 2 \\
X \arrow[ur, "f"] &
\end{tikzcd}
$$

Initial $\leftrightarrow$ final

$$\left(1 \xrightarrow{i} X\right) \quad \xmapsto{2^{(-)}} \quad \left(2^X \xrightarrow{2^i} 2\right)$$

$$\left(X \xrightarrow{f} 2\right) \quad \longleftrightarrow \quad \left(1 \xrightarrow{f} 2^X\right)$$

Reversing the entire automaton

$$\begin{tikzcd}
1 \arrow[dr, "i"] & & 2 \\
& X \arrow[ur, "f"] \arrow[d, "t"] & \\
& X^A &
\end{tikzcd}$$

Reversing the entire automaton

$$
\begin{tikzcd}
1 \arrow[dr, "i"] & & 2 \\
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\end{tikzcd}
\quad \xmapsto{\;2^{(-)}\;} \quad
\begin{tikzcd}
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& 2^X \arrow[ur, "2^i"] \arrow[d, "2^t"] & \\
& (2^X)^A &
\end{tikzcd}
$$

Reversing the entire automaton

$$
\begin{tikzcd}
1 \arrow[dr, "i"] & & 2 & & & 1 \arrow[dr, "f"] & & 2 \\
& X \arrow[ur, "f"] \arrow[d, "t"] & & {} \arrow[r, mapsto, "2^{(-)}"] & {} & & 2^X \arrow[ur, "2^i"] \arrow[d, "2^t"] & \\
& X^A & & & & & (2^X)^A &
\end{tikzcd}
$$

- Initial and final are exchanged . . .

Reversing the entire automaton

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\begin{tikzcd}
1 \arrow[dr, "i"] & & 2 \\
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\end{tikzcd}
\quad \xmapsto{2^{(-)}} \quad
\begin{tikzcd}
1 \arrow[dr, "f"] & & 2 \\
& 2^X \arrow[ur, "2^i"] \arrow[d, "2^t"] & \\
& (2^X)^A &
\end{tikzcd}
$$

- Initial and final are exchanged . . .
- transitions are reversed . . .

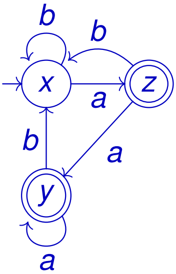
Reversing the entire automaton

$$\begin{tikzcd}
1 \arrow[dr, "i"] & & 2 \\
& X \arrow[ur, "f"] \arrow[d, "t"] & \\
& X^A &
\end{tikzcd}
\quad \overset{2^{(-)}}{\longmapsto} \quad
\begin{tikzcd}
1 \arrow[dr, "f"] & & 2 \\
& 2^X \arrow[ur, "2^i"] \arrow[d, "2^t"] & \\
& (2^X)^A &
\end{tikzcd}$$

- Initial and final are exchanged . . .
- transitions are reversed . . .
- and the result is again deterministic!

Our previous example

$X =$



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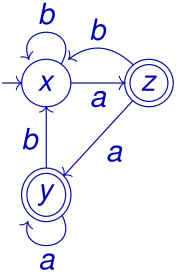
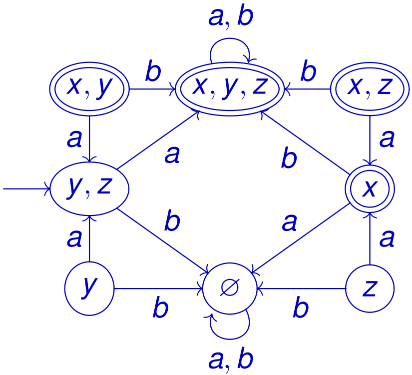


$2^X =$



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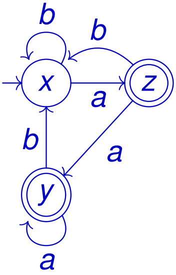
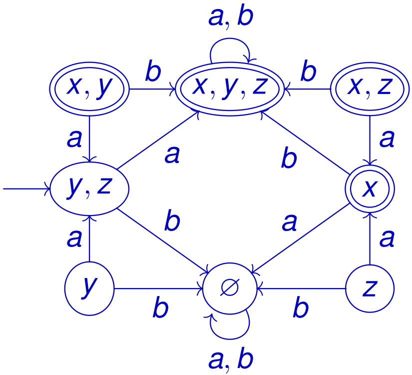


$2^X =$



- Note that X has been reversed and determinized:

$$2^X = det(rev(X))$$

Proving today's Theorem

If: a deterministic automaton X is reachable and accepts $L(X)$

Proving today's Theorem

If: a deterministic automaton X is reachable and accepts $L(X)$

then: 2^X ($=$ $det(rev(X))$) is minimal/observable and

$$L(2^X) = reverse(L(X))$$

Proof: by reversing $A^* \xrightarrow{r} X$

$$\begin{tikzcd}
1 \arrow[d, "\epsilon"'] \arrow[dr, "i"] & \\
A^* \arrow[r, "r"'] \arrow[d, "\alpha"'] & X \arrow[d, "t"] \\
(A^*)^A \arrow[r] & X^A
\end{tikzcd}$$

Proof: by reversing $A^* \xrightarrow{r} X$

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\end{tikzcd}
\qquad \overset{2^{(-)}}{\longmapsto} \qquad
\begin{tikzcd}
 & 2 \\
2^X \arrow[ur, "2^i"] \arrow[r, "2^r"'] \arrow[d, "2^t"'] & 2^{A^*} \arrow[u, "2^\epsilon"'] \arrow[d, "2^\alpha"] \\
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(A^*)^A \arrow[r] & X^A & & & (2^X)^A \arrow[r] & (2^{A^*})^A
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- X becomes 2^X

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\end{tikzcd}
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- initial automaton A^* becomes (almost) final automaton 2^{A^*}

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- X becomes 2^X
- initial automaton A^* becomes (almost) final automaton 2^{A^*}
- r is surjective $\Rightarrow$ 2^r is injective

Reachable becomes observable

$$\begin{tikzcd}
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\quad \overset{2^{(-)}}{\longmapsto} \quad
\begin{tikzcd}
& 2 & \\
2^X \arrow[ur, "2^i"] \arrow[r, "2^r"'] \arrow[d, "2^t"'] & 2^{A^*} \arrow[u, "2^\epsilon"'] \arrow[r, dashed, "rev"'] \arrow[d, "2^\alpha"] & 2^{A^*} \arrow[ul, "\epsilon?"'] \arrow[d, "\beta"'] \\
(2^X)^A \arrow[r] & (2^{A^*})^A \arrow[r, dashed] & (2^{A^*})^A
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& & & 2^X \arrow[uur, "2^i"] \arrow[r, "2^r"'] \arrow[d, "2^t"'] & 2^{A^*} \arrow[uu, "2^\epsilon"'] \arrow[r, dashed, "rev"'] \arrow[d, "2^\alpha"] & 2^{A^*} \arrow[uul, "\epsilon?"'] \arrow[d, "\beta"] \\
(A^*)^A \arrow[r] & X^A & & (2^X)^A \arrow[r] & (2^{A^*})^A \arrow[r, dashed] & (2^{A^*})^A
\end{tikzcd}
$$

- If r is surjective then (2^r and hence) $rev \circ 2^r$ is injective.

Reachable becomes observable

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(2^X)^A \arrow[r] & (2^{A^*})^A \arrow[r, dashed] & (2^{A^*})^A
\end{tikzcd}
$$

- If r is surjective then (2^r and hence) $rev \circ 2^r$ is injective.
- That is, 2^X is observable (= minimal).

Summarizing

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\begin{tikzcd}
1 \arrow[d, "\epsilon"'] \arrow[dr, "i"] & & 2 \\
A^* \arrow[r, dashed, "r"'] \arrow[d, "\alpha"'] & X \arrow[ur, "f"] \arrow[d, "t"'] & \\
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\end{tikzcd}
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1 \arrow[dr, "f"] & & 2 \\
& 2^X \arrow[ur, "2^i"] \arrow[r, dashed, "rev \circ 2^r"'] \arrow[d, "2^t"'] & 2^{A^*} \arrow[u, "\epsilon?"'] \arrow[d, "\beta"] \\
& (2^X)^A \arrow[r, dashed] & (2^{A^*})^A
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- If: X is reachable, i.e., r is surjective

Summarizing

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\begin{tikzcd}
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\end{tikzcd}
\qquad\qquad
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& (2^X)^A \arrow[r, dashed] & (2^{A^*})^A
\end{tikzcd}
$$

- If: X is reachable, i.e., r is surjective

  then: $rev \circ 2^r$ is injective, i.e., 2^X is observable = minimal.

Summarizing

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\begin{tikzcd}
1 \arrow[d, "\epsilon"'] \arrow[dr, "i"] & & 2 \\
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& (2^X)^A \arrow[r, dashed] & (2^{A^*})^A
\end{tikzcd}
$$

- If: X is reachable, i.e., r is surjective

  then: $rev \circ 2^r$ is injective, i.e., 2^X is observable = minimal.
- And: $rev(2^r(f)) = rev(o(i))$, i.e., $L(\, 2^X \,) \;=\; reverse(\, L(X)\,)$

Corollary: Brzozowski's algorithm

- X becomes 2^X, accepting $reverse(L(X))$

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- X becomes 2^X, accepting $\mathit{reverse}(L(X))$
- take reachable part: $Y = \mathit{reachable}(2^X)$

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- X becomes 2^X, accepting $\mathit{reverse}(L(X))$
- take reachable part: $Y = \mathit{reachable}(2^X)$
- Y becomes 2^Y, which is minimal and accepts

$$\mathit{reverse}(\mathit{reverse}(L(X))) = L(X)$$

Generalizations

$$
\begin{tikzcd}
1 \arrow[d, "\epsilon"'] \arrow[dr, "i"] & & 2 \\
A^* \arrow[r, dashed, "r"'] \arrow[d, "\alpha"'] & X \arrow[r, dashed, "o"'] \arrow[d, "t"] \arrow[ur, "f"] & 2^{A^*} \arrow[u, "\epsilon?"'] \arrow[d, "\zeta"] \\
(A^*)^A \arrow[r, dashed, "r^A"'] & X^A \arrow[r, dashed, "o^A"'] & (2^{A^*})^A
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[d, "\epsilon"'] \arrow[dr, "i"] & & B \\
A^* \arrow[r, dashed, "r"'] \arrow[d, "\alpha"'] & X \arrow[r, dashed, "o"'] \arrow[d, "t"] \arrow[ur, "f"] & B^{A^*} \arrow[u] \arrow[d] \\
(A^*)^A \arrow[r, dashed, "r^A"'] & X^A \arrow[r, dashed, "o^A"'] & (B^{A^*})^A
\end{tikzcd}
$$

- A Brzozowski minimization algorithm for Moore automata.

$$B^X = \{\varphi \mid \varphi\colon X \to B\} \qquad B^f(\varphi) = \varphi \circ f$$

Further generalizations

- Moore automata generalization: uniform algorithm for decorated traces and must testing (joint work with Bonchi, Caltais and Pous);
- Further generalizations to non-deterministic and weighted automata.

Conclusions

- Abstract analysis can bring new perspectives/results.
- Combination algebra-coalgebra is fruitful.
- (Co)algebra is not only semantics but also algorithms!

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Thanks!